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AUTOMORPHISMS OF THE COMPLEX NUMBERS

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One of the best known bits of mathematical folklore is that there are many automorphisms of the field of complex numbers, i.e., that the complex numbers can be permuted in many ways (besides the familiar complex conjugation) that preserve addition and multiplication. As evidence that it is folklore we point out that it appears without proof in a popular projective geometry text [6] as well as an undergraduate algebra text [5]. This expository paper is devoted to a proof of this bit of folklore. The average mathematician is vaguely aware that the "wild" automorphisms of the complex number system probably require for their construction the axiom of choice or some equivalent assumption about sets. In our existence proof for wild automorphisms we illustrate a typical application of Zorn's lemma; moreover, we present evidence (Theorem 4) that wild automorphisms are so wild that an assumption such as Zorn's lemma or the axiom of choice seems to be essential.

1. Subfields of the field of complex numbers. Any field considered in this paper will be a subfield of the complex numbers, **C**, i.e., will be a subset of **C** containing 0, 1 and containing $a+b$, ab, $a-b$, and (if $b\neq 0$) a/b whenever it contains a and b. Familiar examples of subfields are **Q**, the field of rational numbers; **R**, the field of real numbers; and **C** itself. It is easy to show that the intersection of any collection of subfields of **C** is itself a subfield of **C**, and, in particular, that the intersection of all subfields is **Q**.

DEFINITIONS. Let **F** be a subfield of **C** and let $\alpha, \beta, \cdots, \lambda$ be complex numbers. The intersection of all subfields of **C** containing **F** and $\alpha, \beta, \cdots, \lambda$ is denoted by $\mathbf{F}(\alpha, \beta, \cdots, \lambda)$ and is called the *extension field* of **F** *generated* by $\alpha, \beta, \cdots, \lambda$. The numbers $\alpha, \beta, \cdots, \lambda$ are called *generators*. If **G** is a subfield of **C** containing **F** such that $\mathbf{G}=\mathbf{F}(\alpha, \beta, \cdots, \lambda)$ for some finite set of generators, then we say that **G** is a *finitely generated extension* of **F**. If only one generator is required then we say that **G** is a *simple extension* of **F**.

A complex number, α, is called *algebraic* or *transcendental* over **F** according as it does or does not satisfy at least one polynomial equation with coefficients in **F**. If α is algebraic over **F** then the (unique!) monic polynomial, p, of least degree with coefficients in **F** such that $p(\alpha)=0$ is called the *minimal polynomial* of α over **F**.

The structure of a simple extension, $\mathbf{F}(\alpha)$, of **F** depends on the "algebraic relationship" between α and **F**. If there is none, i.e., if α is transcendental over **F**, then distinct rational forms, $p(x)/q(x)$ (p, q polynomials with coefficients in **F**) yield distinct complex numbers, $p(\alpha)/q(\alpha)$, all in $\mathbf{F}(\alpha)$. However if α is algebraic over **F**, and if m is the degree of its minimal polynomial over **F**, then one can show that $q(\alpha)\neq 0$ implies $p(\alpha)/q(\alpha)=a_0+a_1\alpha+\cdots+a_{m-1}\alpha^{m-1}$ for exactly one set of $a_i \in \mathbf{F}$. In any case $\mathbf{F}(\alpha)=\{p(\alpha)/q(\alpha) \mid p, q$ polynomials with coefficients in **F** and $q(\alpha)\neq 0\}$. For details see [2], Chapter 14, Theorems 1 and 4.

DEFINITIONS. A subfield, **F**, of **C** is said to be *algebraically closed* if every complex number algebraic over **F** is in **F**, or, equivalently, if every polynomial with coefficients in **F** can be factored into linear factors with coefficients in **F**. We shall denote by $\mathbf{F}^a$ the set of all complex numbers which are algebraic over **F**. $\mathbf{F}^a$ is called the *algebraic closure* of **F** in **C**.

THEOREM 1. *Let* **F** *be a subfield of* **C**. $\mathbf{F}^a$ *is a subfield of* **C** *that is algebraically closed.*

There are two "standard" proofs of this theorem. In the first proof one shows that any rational combination of numbers algebraic over **F** is itself algebraic over **F**. To see that this is not easy the reader should try to prove that $\alpha+\beta$ and $\alpha\beta$ are algebraic over **F** whenever α and β are. The second proof depends on the fact that a field is also a vector space over any of its subfields. The two main lemmas for this proof are a multiplicative property of dimensions for extension fields and the fact that α is algebraic over **F** if and only if $\mathbf{F}(\alpha)$ is a *finite dimensional* vector space over **F**. Both proofs are in [3].

2. Isomorphisms between fields. Roughly speaking an isomorphism between fields is a one to one correspondence between the elements of the two fields which "preserves" algebraic operations. Since we shall be concerned with isomorphisms between subfields of the complex number system and because we plan to apply Zorn's lemma to sets of isomorphisms we choose the following specialized definition of isomorphism.

DEFINITIONS. An *isomorphism* (between subfields of **C**) is a set, ϕ, of ordered pairs of complex numbers such that: 1. If $\langle a, x\rangle$ and $\langle b, y\rangle$ are in ϕ, i.e., if $\phi(a)=x$ and $\phi(b)=y$, then $a=b$ if and only if $x=y$. (In other words, ϕ is a function and is one to one.) 2. If $\langle a, x\rangle$ and $\langle b, y\rangle$ belong to ϕ then so do $\langle a+b, x+y\rangle$, $\langle ab, xy\rangle$, $\langle a-b, x-y\rangle$, and (if b and y are nonzero) $\langle a/b, x/y\rangle$. (ϕ preserves algebraic operations.) 3. $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$ are both in ϕ. (This assures that ϕ is not trivial in the sense of being empty or containing only $\langle 0, 0\rangle$.)

As is customary we call the set of all first components of ordered pairs in an isomorphism ϕ the *domain* of ϕ, and the set of all second components the *range* of ϕ. Also, as usual, we write $\phi(a)=x$ and say ϕ *sends* a to x if and only if $\langle a, x\rangle \in\phi$. It is easy to show that the domain and range of an isomorphism are subfields of **C**. If the domain and range are the same field, **F**, then we say that ϕ is an *automorphism* of **F**. Clearly the identity map on a subfield **F**, $I_F=\{\langle x, x\rangle \mid x \in \mathbf{F}\}$, is an automorphism of **F**. This is called the *trivial* automorphism of **F**. All other automorphisms of **F** are called *nontrivial.*

Let ϕ and σ be two isomorphisms. We say that ϕ *extends* σ if σ is a subset of ϕ. If, in addition, the domain of ϕ is **F** then we say that ϕ *extends* σ *to* **F**.

Caution! Our use of the word isomorphism is very restricted in that we allow only complex numbers in the domain and range. The usual definition of "isomorphism" (between arbitrary fields) is more complicated in that not only are more general "numbers" allowed but also the operations in the two fields involved may be different. The reader who is familiar with a different definition of isomorphism should prove that if all fields involved are subfields of **C** then the definition above is equivalent to his "standard" definition.

Examples of isomorphisms. The most familiar example of an isomorphism is complex conjugation, $\{\langle a+bi, a-bi\rangle \mid a, b\in\mathbf{R}\}$, which is a nontrivial automorphism of **C**. Slightly more complicated examples are $\sigma=\{\langle a+c\sqrt{7}, a-c\sqrt{7}\rangle \mid a, c \in\mathbf{Q}\}$ and $\psi=\{\langle a+b\sqrt[4]{7}+c\sqrt{7}+d\sqrt[4]{343},\ a+ib\sqrt[4]{7}-c\sqrt{7}-id\sqrt[4]{343}\rangle \mid a, b, c, d \in\mathbf{Q}\}$. The reader should verify that σ is an automorphism of $\mathbf{Q}(\sqrt{7})$, that ψ extends σ to $\mathbf{Q}(\sqrt[4]{7})$, and that the range of ψ is $\mathbf{Q}(i\sqrt[4]{7})$.

THEOREM 2. *Any isomorphism between subfields of* **C** *extends* $I_{\mathbf{Q}}$, *the identity map on* **Q**.

Proof. Let ϕ be an isomorphism and let $\mathbf{F}=\{a \mid \phi(a)=a\}=\{a \mid \langle a, a\rangle\in\phi\}$. It is easy to show that **F** is a subfield of **C**. Since **Q** is contained in any subfield, ϕ must extend $I_{\mathbf{Q}}$.

This result asserts that the field of rational numbers is an "algebraically rigid" structure. This is not too surprising since 0 and 1 can be characterized algebraically, (0 as the only solution of $x+x=x$ and 1 as the only nonzero solution of $xx=x$) and all other rational numbers are built up by rational operations from 0 and 1. Thus if a function is to preserve algebraic properties it should leave the rational numbers undisturbed. A similar (but more surprising) result is valid for the field of real numbers.

THEOREM 3. *The only isomorphisms between subfields of* **C** *whose domains include* **R** *and which map* **R** *into* **R** *are* $I_{\mathbf{R}}$, $I_{\mathbf{C}}$, *and complex conjugation.*

Proof. Let ϕ be such an isomorphism, i.e., assume $\mathbf{R}\subseteq$domain ϕ and $x\in\mathbf{R}$ implies $\phi(x)\in\mathbf{R}$. We first show that ϕ preserves order in **R**. If $x<y$, then there is a real number w such that $w\neq 0$ and $y-x=w^2$. But then $\phi(y)-\phi(x)=[\phi(w)]^2$ with $\phi(w)\in\mathbf{R}$ and $\phi(w)\neq 0$. Hence $\phi(y)-\phi(x)$ is positive, i.e., $\phi(x)<\phi(y)$. Now assume $a\in\mathbf{R}$, but that $a\neq\phi(a)$. Choose a rational number, q, between a and $\phi(a)$. Since $\phi(q)=q$ by Theorem 2, the order between a and q is reversed by ϕ and we have a contradiction. Hence $a\in\mathbf{R}$ implies $\phi(a)=a$, i.e., $I_{\mathbf{R}}\subseteq\phi$.

If $\phi\neq I_{\mathbf{R}}$, then the domain of ϕ is a subfield of **C** containing **R** as a proper subset. In any such subfield we can find a complex number, $a+bi$, with $b\neq 0$ as well as all real numbers. But, since $x+yi=x+y([(a+bi)-a]/b)$, this implies that the subfield is **C** itself. Thus the domain of ϕ is **C**. Consider $\phi(i)$. Since $i^2=-1$, $[\phi(i)]^2=\phi(-1)=-1$. The only roots of $x^2=-1$ are $\pm i$; hence $\phi(i)=\pm i$. If $\phi(i)=i$, then $\phi=I_{\mathbf{C}}$, and if $\phi(i)=-i$, then ϕ is complex conjugation.

Theorem 2 implies that **Q** has no nontrivial automorphism, and Theorem 3 implies the same for **R**. Theorem 3 also implies that a nontrivial automorphism of a subfield of **R** cannot be extended to an automorphism of **R**. For example, the automorphism σ defined just before Theorem 2 cannot be extended to an automorphism of **R**.

We shall call any automorphism of **C** which is not $I_{\mathbf{C}}$ nor complex conjugation a *wild automorphism* of **C**. That these automorphisms are really "wild" is shown by the following theorem.

THEOREM 4. *If ϕ is a wild automorphism of* **C** *then ϕ is a discontinuous mapping of the complex plane onto itself; in fact, ϕ leaves a dense subset of the real line pointwise fixed but maps the real line onto a dense subset of the plane.*

Proof. By Theorem 2, ϕ leaves **Q** (a dense subset of the real line!) pointwise fixed. By Theorem 3 we can choose $b \in \mathbf{R}$ such that $\phi(b) \notin \mathbf{R}$. Every neighborhood of b contains a rational number (which is left fixed by ϕ) and the number b (which is moved by ϕ); hence ϕ is discontinuous.

For every pair of rational numbers, q and r, $\phi(rb+q)=\phi(r)\phi(b)+\phi(q) = r\phi(b)+q$. Thus for a fixed r, $\{\phi(rb+q) \mid q \in \mathbf{Q}\}$ is a set of images of real numbers which is a dense subset of the horizontal line through $r\phi(b)$. As r varies this horizontal line moves up and down; moreover the various $r\phi(b)$ form a dense subset of the (nonhorizontal) line through 0 and $\phi(b)$. Thus the set $\{\phi(rb+q) \mid r, q \in \mathbf{Q}\}$ is a dense subset of the plane. This set is contained in $\phi(\mathbf{R})$; hence $\phi(\mathbf{R})$ is also a dense subset of **C**.

3. Isomorphisms and simple extensions. As a first step in extending an automorphism to an automorphism of **C** we need to know how to extend it to a "slightly larger" subfield. Since the proofs are no harder we discuss the more general topic of extending an isomorphism to a simple extension of its domain. There are two cases to consider according as the generator of the simple extension is algebraic or transcendental over the original field.

THEOREM 5A. *Let ϕ be an isomorphism with domain* **F** *and range* **F**$'$. *If α is algebraic over* **F** *then there is an isomorphism extending ϕ to* **F**(α) *and sending α to β if and only if β is a root of the polynomial obtained by applying ϕ to the coefficients of the minimal polynomial of α over* **F**.

THEOREM 5B. *Let ϕ be an isomorphism with domain* **F** *and range* **F**$'$. *If α is transcendental over* **F**, *then there is an isomorphism extending ϕ to* **F**(α) *and sending α to β if and only if β is transcendental over* **F**$'$.

An outline of the proof. In either case it is easy to show that $\sigma = \{\langle p(\alpha)/q(\alpha), p'(\beta)/q'(\beta)\rangle \mid p, q$ are polynomials with coefficients in **F**, $q(\alpha) \neq 0$, p', q' obtained by applying ϕ to the coefficients of p and $q\}$ is the only possible isomorphism extending ϕ to **F**(α) and sending α to β. It is tedious, but not difficult, to show that σ is an isomorphism if and only if α and β are related as stated in Theorems 5A or 5B. For more details of this proof see [2], Chapter 14, Theorem 1 and Chapter 15, Lemma 1.

A special case of Theorem 5A comprises part of the proof of Theorem 3. In that proof we showed that the only extensions of I_R to $\mathbf{R}(i)=\mathbf{C}$ send i to $\pm i$.

If we combine Theorems 5A and B we find that any isomorphism with domain **F** and range **F**$'$ can be extended to **F**(α) *unless* α is transcendental over **F** and there are no complex numbers transcendental over **F**$'$. We shall show at the end of the paper that this "unless" clause is an essential qualification.

Examples. Let ψ and σ be the isomorphisms defined just before Theorem 2. By Theorem 5A the only extensions of I_Q to $\mathbf{Q}(\sqrt{7})$ are σ and the identity map on $\mathbf{Q}(\sqrt{7})$ since $\sqrt{7}$ and $-\sqrt{7}$ are the only two roots in **C** of the polynomial

x^2-7. The minimal polynomial of $\sqrt[4]{7}$ over $\mathbf{Q}(\sqrt{7})$ is $x^2-\sqrt{7}$ which is sent by σ to $x^2+\sqrt{7}$. The only two roots of $x^2+\sqrt{7}$ are $\pm i\sqrt[4]{7}$; hence an extension of σ to $\mathbf{Q}(\sqrt[4]{7})$ must send $\sqrt[4]{7}$ to one of these two numbers. Thus there are only two possible extensions of σ to $\mathbf{Q}(\sqrt[4]{7})$, one of which is ψ.

There are uncountably many complex numbers which are transcendental over the range of ψ, $\mathbf{Q}(i\sqrt[4]{7})$. Thus by Theorem 5B there are uncountably many ways of extending ψ to $\mathbf{Q}(\sqrt[4]{7}, \pi)$. A few of these possibilities send π to $1/\pi$, $1-\pi$, $\pi+\sqrt{57}$, or $e/17$.

These examples should convince the reader that there are many isomorphisms between finitely generated extensions of $\mathbf{Q}$. Since many of these are clearly automorphisms differing radically in their action from I_C or complex conjugation, it will follow from our main result (any automorphism can be extended to an automorphism of $\mathbf{C}$) that there are many wild automorphisms of $\mathbf{C}$.

Using ordinary induction and Theorems 5A and B we could extend any automorphism of a field to a finitely generated extension of that field. Unfortunately $\mathbf{C}$ is not a finitely (or even countably) generated extension of $\mathbf{Q}$ so ordinary induction will not suffice to prove that *any* automorphism of a subfield of $\mathbf{C}$ can be extended to $\mathbf{C}$. We therefore pause to discuss a tool to handle the "transfinite" aspect of our induction.

4. Zorn's lemma. A nonempty collection, $\mathfrak{C}$, of sets is called a *chain* of sets if for any two sets A, B in $\mathfrak{C}$, either $A\subseteq B$ or $B\subseteq A$. A family, $\mathfrak{F}$, of sets is said to have the *chain property* if $\mathfrak{F}$ contains the union of every chain of sets taken from $\mathfrak{F}$. Since the union of any finite chain of sets is simply the largest set in that chain, it is clear that any finite family of sets has the chain property. Two more examples of families with the chain property are $\mathfrak{F}_1=$ the set of all subsets of a given set A, and $\mathfrak{F}_2=\{B \mid B\subseteq\mathbf{R}$ and B contains no integers$\}$. Two families without the chain property are $\mathfrak{G}_1=\{A \mid A$ is a finite subset of $\mathbf{R}\}$ and $\mathfrak{G}_2=\{\mathbf{F} \mid \mathbf{F}$ is a subfield of $\mathbf{C}$ and a finitely generated extension of $\mathbf{Q}\}$.

ZORN'S LEMMA. *If $\mathfrak{F}$ is a nonempty family of subsets of a given set B and $\mathfrak{F}$ has the chain property, then there is at least one set, M, in $\mathfrak{F}$ such that $A\in\mathfrak{F}$ and $M\subseteq A$ implies $M=A$.*

A set with the property specified for M is called a *maximal* set in $\mathfrak{F}$. It is quite possible for a family of sets to have many maximal elements. Think of them as located at the tips of branches rather than at the top of the heap. Zorn's lemma only requires that under certain conditions there must be at least one maximal element. Several other properties of sets, notably the axiom of choice, are equivalent to Zorn's lemma. For a discussion of these equivalences (including proofs) see [4] or [7]. Returning to the examples of families with the chain property we note that the maximal set in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ is unique. The reader should have no difficulty constructing a finite family of sets in which there is more than one maximal set. Neither of the families $\mathfrak{G}_1$ or $\mathfrak{G}_2$ has a maximal element. The family $\mathfrak{H}=\{A \mid$ Either A is a finite subset of $\mathbf{Q}$ or $A=\mathbf{Q}\}$ is an example of a family of sets which has a maximal member but does not satisfy the chain property; hence the converse of Zorn's lemma is not true.

5. Extending automorphisms to C. We now show that any automorphism, ϕ, can be extended to **C** by applying Zorn's lemma to the family of automorphisms extending ϕ. It is awkward to do this directly since the only isomorphisms extending ϕ to a simple extension of its domain may not be automorphisms. (Consider our examples σ and ψ!) To avoid this difficulty we first prove the following theorem.

THEOREM 6. *If ϕ is an isomorphism with domain* **F** *and range* **G**, *then ϕ can be extended to an isomorphism with domain* $\mathbf{F}^a$ *and range* $\mathbf{G}^a$.

Proof. Let $\mathfrak{F}=\{\theta|\theta$ is an isomorphism extending ϕ to a subfield of $\mathbf{F}^a\}$. We shall show that $\mathfrak{F}$ satisfies the three hypotheses of Zorn's lemma. $\mathfrak{F}$ is nonempty since ϕ extends itself to **F**. Isomorphisms are sets of ordered pairs; hence all members of $\mathfrak{F}$ are subsets of $\mathbf{C}\times\mathbf{C}$. Let $\mathfrak{C}$ be a chain taken from $\mathfrak{F}$ and let σ be the union of all θ in $\mathfrak{C}$. σ is clearly a set of ordered pairs of complex numbers. $\mathfrak{C}$, as a chain, is nonempty; hence it contains at least one isomorphism and thus $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$ are in σ. Let $\langle a, b\rangle$ and $\langle x, y\rangle$ be in σ. Then $\langle a, b\rangle\in\theta_1$ and $\langle x, y\rangle\in\theta_2$ for some $\theta_1, \theta_2\in\mathfrak{C}$. Since $\mathfrak{C}$ is a chain either $\theta_1\subseteq\theta_2$ or $\theta_1\supseteq\theta_2$ and thus the two ordered pairs are both in the larger of θ_1 and θ_2. From this it follows easily that σ is a one to one function which preserves algebraic operations. The isomorphism σ is in the family $\mathfrak{F}$ since it clearly extends ϕ and its domain, the union of subfields of $\mathbf{F}^a$, is contained in $\mathbf{F}^a$. We apply Zorn's lemma and let ψ be a maximal member of $\mathfrak{F}$. We must show that the domain and range of ψ are $\mathbf{F}^a$ and $\mathbf{G}^a$.

If the domain of ψ is not all of $\mathbf{F}^a$, then there is at least one element α in $\mathbf{F}^a$ but not in the domain of ψ. Since α is algebraic over **F** and $\mathbf{G}^a$ is algebraically closed there is at least one β in $\mathbf{G}^a$ which is a root of the ψ transform of the minimal polynomial of α over **F**. Thus by Theorem 5A there is at least one way of extending ψ to a larger isomorphism still in $\mathfrak{F}$. This is a contradiction and thus $\mathbf{F}^a$ is the domain of ψ.

Since $\mathbf{F}^a$ is algebraically closed and ψ is an isomorphism, the range of ψ is an algebraically closed subfield of $\mathbf{G}^a$ which contains **G**. But the only such subfield of $\mathbf{G}^a$ is $\mathbf{G}^a$ itself; hence $\mathbf{G}^a$ is the range of ψ and the proof is complete.

THEOREM 7. *Any automorphism of a subfield of* **C** *can be extended to an automorphism of* **C**.

Proof. Let ϕ be an automorphism of a subfield of **C**, and let $\mathfrak{F}=\{\theta|\theta$ is an automorphism extending ϕ to some subfield of $\mathbf{C}\}$. The proof that $\mathfrak{F}$ satisfies the three hypotheses of Zorn's lemma is virtually the same as in the proof of Theorem 6, the only change necessary is to show that domain $\sigma=$range σ instead of domain $\sigma\subseteq\mathbf{F}^a$. We leave this to the reader. Applying Zorn's lemma let ψ be a maximal member of $\mathfrak{F}$. We must show domain $\psi=\mathbf{C}$. If not, then there is a complex number, α, not in domain $\psi=\mathbf{F}$. If α is algebraic over **F** then, by Theorem 6, we could extend ψ to an automorphism of $\mathbf{F}^a$ contradicting the maximality of ψ in $\mathfrak{F}$. If α is transcendental over **F**, then by Theorem 5B we could extend ψ to an automorphism of $\mathbf{F}(\alpha)$, sending α to α for example, since α is also transcendental over range $\psi=\mathbf{F}$. This again contradicts the maximality of ψ, so there can be no complex numbers outside of domain ψ and the proof is complete.

6. Concluding remarks.

1. Although it is doubtful that anyone will give a complete recipe for an automorphism of **C** aside from I_C or complex conjugation, we see from the Theorem above that any automorphism that can be constructed in a finitely generated extension of **Q** can be extended to **C**. Thus, for example, there are automorphisms of **C** which interchange π and e, send $\sqrt[4]{3}$ to $i\sqrt[4]{3}$, and leave $\sqrt{7}$ fixed.

2. It is not true that any isomorphism between subfields of **C** can be extended to an automorphism of **C**. In particular there *are* isomorphisms with domain **C** whose range is properly contained in **C**. For example, choose $\alpha_1, \alpha_2, \alpha_3, \cdots$, a countable set of complex numbers that are algebraically independent over **Q**. There is an isomorphism, ϕ, of $\mathbf{Q}(\alpha_1, \alpha_2, \cdots)$ into itself such that $\phi(\alpha_i) = \alpha_{i+1}$. Applying Zorn's lemma to $\mathfrak{F} = \{\theta | \theta$ is an isomorphism extending ϕ, range $\theta \subseteq$ domain θ, and α_1 transcendental over range $\theta\}$ leads to a maximal isomorphism, ψ, whose domain is all of **C** but such that α_1 is not in the range. Note that ψ^{-1} is an example of an isomorphism defined on a subfield, **F**, of **C** which cannot be extended to $\mathbf{F}(\alpha_1)$.

3. As the final comment I mention an additional bit of mathematical folklore. In [1] it is claimed, without proof or reference to the proof, that the cardinality of the set of automorphisms of **C** is $2^{2^{\aleph_0}}$. I have heard this from other sources and am convinced that it is true although I do not know where the proof may be found.

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ON SOLUTIONS OF CERTAIN RICCATI DIFFERENTIAL EQUATIONS

JAMES S. W. WONG, University of Alberta, Edmonton

In search of exact solutions of the general Riccati differential equation

$$y' = f + gy + hy^2, \tag{1}$$

where the differentiation is with respect to x and f, g, h are functions of x, it is customary to find conditions on the coefficients f, g and h such that equation (1) may be transformed into another first order equation where the variables

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where the differentiation is with respect to x and f, g, h are functions of x, it is customary to find conditions on the coefficients f, g and h such that equation (1) may be transformed into another first order equation where the variables are separable. Two of such conditions were given in recent articles by Rao [1] and Allen and Stein [2]. The purpose of the present note is to present a slightly more general approach which includes both of the above-mentioned conditions and also serves to reveal the interplay between the results of [1] and [2].

Consider the following transformation

$$y = uv - w, \tag{2}$$

where v and w are to be determined in terms of f, g and h. Substituting (2) into (1), we obtain another Riccati equation in the variable u,

$$u' = F + Gu + Hu^2, \tag{3}$$

where $F=(f-gw+w'+hw^2)/v$, $G=(gv-v'-2vwh)/v$, and $H=hv$. In the transformed equation (3) the variables can be separated if there exist constants α, β such that $F=\alpha H$ and $G=\beta H$. In other words, there exist solutions v and w for the following simultaneous first order equations

$$\begin{aligned} w' - gw + hw^2 + f &= \alpha hv^2, \\ v' - gv + 2whv &= -\beta hv^2. \end{aligned} \tag{4}$$

In general, it is difficult to find the general solution of (4). However, we note that any particular solution of (4) will lead to a general solution of equation (1). Thus, by using different sets of particular solutions of (4), we are able to derive various conditions involving f, g, h which guarantee that the variables in equation (3) can be separated. In case of [1], we take $w=g/h$ and $v^2=W/\alpha h^3$, where $W=fh^2-gh'-hg'$. In order that such v and w be solutions to (4), we must have the following condition:

$$\frac{hW' - 3h'W - 2ghW}{2W^{3/2}h^{1/2}} = -\frac{\beta}{\sqrt{\alpha}}\,. \tag{5}$$

Similarly in [2], we take $w=0$ and $v=f/g$. In this case, the condition to be satisfied is:

$$\frac{g + \dfrac{h'}{2h} - \dfrac{f'}{2f}}{\sqrt{fh}} = \frac{\beta}{\sqrt{\alpha}}\,. \tag{6}$$

Conditions (5) and (6) are exactly those given in [1] and [2], which appeared to be obtained there merely by guessing. Here, we can see that they arise in order to guarantee the existence of certain particular solutions of (4). Through (5) and (2), we obtain the following first order equation in which it is easy to see that the variables can be separated

$$u' = \left(\frac{W}{h}\right)^{1/2}\left(1 + \frac{\beta}{\sqrt{\alpha}}\,u + u^2\right). \tag{7}$$

Similarly for condition (6), we arrive at the following equation:

$$u' = \sqrt{fh}\left(1 + \frac{\beta}{\sqrt{\alpha}}u + u^2\right). \tag{8}$$

By choosing particular solutions v and w of appropriate forms (in terms of f, g and h), one can generate various conditions analogous to those given by (5) and (6). Although there does not seem to exist a general method (other than by trial and error) for choosing v and w, one may easily find useful conditions similar to (5) and (6) by this approach. For example, if we choose $v=\exp(\int^x g)$ and $w=-\beta/2\exp(\int^x g)$, where β is some nonnegative constant such that

$$\frac{f}{h}\exp\left(-2\int^x g\right) = \left(\alpha - \frac{\beta^2}{4}\right) \tag{9}$$

then equation (3) reads

$$u' = h\left[\exp\left(\int^x g\right)\right](\alpha + \beta u + u^2), \tag{10}$$

where the variables can be separated.

As an example, we consider the following simple Riccati equation

$$y' = xe^{-2x} - y + xy^2 \tag{11}$$

defined for $0 \leqq x \leqq 1$ with the initial condition $y(0)=0$. It can readily be verified that the coefficients of (11) satisfy condition (9) with $\alpha=1$ and $\beta=0$, but do not satisfy either (5) or (6). In this case, $y=uv$, and equation (3) becomes

$$u' = xe^{-x}(1 + u^2) \tag{12}$$

which is directly integrable. Since $y(0)=0$ and $v(0)=1$, we have $u(0)=0$. From (12), we obtain the following solution to equation (11):

$$y = e^{-x}\tan(1 - e^{-x}(1 + x)).$$

References

1. P. R. P. Rao, The Riccati differential equation, Amer. Math. Monthly, 69 (1962) 995.

2. J. L. Allen and F. M. Stein, On solutions of certain Riccati differential equations, Amer. Math. Monthly, 71 (1964) 1113–1115.

NOTE ON NEWTON'S METHOD

L. M. WEINER, Illinois Teachers College Chicago-North

When the method of successive approximations is used to approximate a numerical value, it should be ascertained, as part of the procedure, whether each value obtained is actually an improvement on the preceding one. In applying Newton's Method to an equation $f(x)=0$, for example, one starts with an approximation x_1 to a root of $f(x)$ and then uses the formula

$$x_2 = x_1 - y_1/y_1' \tag{1}$$

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$$x_2 = x_1 - y_1/y_1' \tag{1}$$

with $y_1=f(x_1)$ to obtain a second, supposedly better, approximation.

If a is the root sought after, the following two questions might occur: (I) How can one determine whether x_2 is a better or worse approximation than x_1? and (II) Is it possible to construct a function $y=f(x)$ with continuous derivative for $-\infty<x<\infty$ for which the approximations obtained by (1) become successively worse regardless of the choice of x_1?

The latter question asks whether it is possible to have

$$|x_1 - y_1/y_1' - a| > |x_1 - a| \tag{2}$$

or dropping subscripts,

$$|(x-a) - y/y'| > |x-a| \tag{3}$$

for all values of $x (x \neq a)$ and $y=f(x)$.

The inequality (3) can be seen to be true only if one of the following conditions holds:

(i) $(x-a)>0, \quad y/y'<0,$

(ii) $(x-a)>0, \quad y/y'>0, \quad |y/y'|>2|x-a|,$

(iii) $(x-a)<0, \quad y/y'>0,$

(iv) $(x-a)<0, \quad y/y'<0, \quad |y/y'|>2|x-a|.$

Case (i) is clearly impossible since y and y' cannot have opposite signs for all $x>a$. This would imply that if $y>0$, y is monotonic decreasing for $x>a$ and would contradict the assumption that $y>0$ since $y=0$ at $x=a$. The assumption $y<0$ leads to a similar contradiction. Similarly, it may be shown that case (iii) is impossible.

The assumption that $|y/y'|>2|x-a|$ in case (ii) implies

$$|y'| < \tfrac{1}{2}\left|\frac{y}{x-a}\right| \tag{4}$$

for all $x>a$. However,

$$\lim_{x\to a} \tfrac{1}{2}[y/(x-a)] = \tfrac{1}{2} \lim_{x\to a} [f(x)/(x-a)] = \tfrac{1}{2}f'(a);$$

whereas $\lim_{x\to a} y'=f'(a)$ which contradicts (4). Case (iv) leads to a similar contradiction.

Thus the answer to the second question above is in the negative. It is, of course, possible to have a function where one can find a specific value of x_1 which will give successively worse approximations.

If one changes the inequality in (3) to an equality one obtains

$$|(x-a) - y/y'| = |x-a|. \tag{5}$$

Eliminating the possibility that $y/y'=0$ in which we are not interested here, yields the differential equation

$$y' = \tfrac{1}{2}[y/(x-a)] \tag{6}$$

which leads to

$$y = c\sqrt[4]{(x - a)^2}. \tag{7}$$

This function has the interesting property that $(x_2-a) = -(x_1-a)$ regardless of the choice of x_1; i.e., Newton's Method simply will not work here.

The answer to the first question generally requires a knowledge of the behavior of $f(x)$ between $x=x_1$ and $x=a$, and we cannot answer the question in terms of x_1, y_1, and y_1' alone. We may, however, state a necessary and sufficient condition that x_2 be a better approximation than x_1 in terms of x_1, y_1, y_1', and a.

Assuming the function $y=f(x)$ to be given and considering (5) as an equation to be solved for x, we obtain

$$y/y' = 2(x - a) \tag{8}$$

since y/y' and $(x-a)$ must have the same sign.

This is an expression for what we might call the critical values of x since when y/y' and $(x-a)$ have the same sign, $|y/y'| < 2|x-a|$ implies that $|(x-a)-y/y'| < |x-a|$, and Newton's Method leads to a better approximation; $|y/y'| > 2|x-a|$ implies that $|(x-a)-y/y'| > |x-a|$, and Newton's Method leads to a worse approximation; $|y/y'| = 2|x-a|$ implies that $|(x-a)-y/y'| = |x-a|$, and Newton's Method leads to an approximation which is neither better nor worse than the original. When y/y' and $(x-a)$ have opposite signs, $|(x-a)-y/y'| > |x-a|$, and Newton's Method leads to a worse approximation. Of course, it is possible for x_2 to be better than x_1 but for x_3 to be worse than x_2.

The function defined by

$$f(x) = \begin{cases} \dfrac{1 - \sqrt{(1 + 4x^2)}}{2x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}$$

for example, is continuous and has critical values $x = \pm\sqrt{3}/2$ if a is taken to be 0. For this function, if one picks $|x_1| < \sqrt{3}/2$, the x_i will become successively better; if one picks $|x_1| > \sqrt{3}/2$, the x_i will become successively worse; if one picks $|x_1| = \sqrt{3}/2$, the x_i will alternate between $\sqrt{3}/2$ and $-\sqrt{3}/2$.

THE ANALYTICITY OF THE ROOTS OF A POLYNOMIAL AS FUNCTIONS OF THE COEFFICIENTS

DAVID R. BRILLINGER, The London School of Economics and Political Science

1. Introduction. Although it was recognized many years ago [4] that the roots of a polynomial are continuous functions of the coefficients, the fact that they are in addition analytic functions in certain component domains does not appear in the literature. This note presents a theorem on the analyticity of the roots of the complex polynomial $P(z) = z^n + a_1z^{n-1} + \cdots + a_n$ as functions of $(a_1, \cdots, a_n)$.

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2. Analyticity. Unless otherwise specified, in this paper C will denote the complex plane, a region will denote an open set in C^n, while a domain will denote an open and connected set in C^n.

The following theorem is a trivial extension of a theorem of Bochner and Martin, ([2], p. 39).

THEOREM. *If the functions*

$$F_j(b_1, \cdots, b_k; a_1, \cdots, a_n) \qquad j = 1, \cdots, k$$

are analytic functions of $k+n$ variables in a neighborhood of $(w_1, \cdots, w_k; z_1, \cdots, z_n)$, if $F(w_1, \cdots, w_k; z_1, \cdots, z_n) = 0$, and if

$$\left.\frac{\partial(F_1, \cdots, F_k)}{\partial(b_1, \cdots, b_k)}\right|_{b=w,a=z} \neq 0,$$

then the equations

$$F_j(b_1, \cdots, b_k; a_1, \cdots, a_n) = 0 \qquad j = 1, \cdots, k$$

have a unique solution $b_j = b_j(a_1, \cdots, a_n)$ equal to w_j for $a = z$ and analytic in a neighborhood of $(z_1, \cdots, z_n)$.

Returning to the problem at hand we let the roots of $P(z)$ be $(b_1, \cdots, b_n)$; then for $j = 1, \cdots, n$

$$\sum_{i=1}^{n} b_i^j = Q_j(a_1, \cdots, a_n), \tag{1}$$

where Q_j is a polynomial in the a's for each j (see [3], p. 91, for example).

Defining

$$F_j(b_1, \cdots, b_n; a_1, \cdots, a_n) = \sum_{i=1}^{n} b_i^j - Q_j(a_1, \cdots, a_n)$$

for $j = 1, \cdots, n$ we see that F_j is analytic everywhere; also up to a constant multiplier

$$\left.\frac{\partial(F_1, \cdots, F_n)}{\partial(b_1, \cdots, b_n)}\right|_{b=w,a=z}$$

is given by the Vandermonde determinant

$$\text{Det}\,\|b_i^{j-1}\| = \prod_{1 \leq i < j \leq n} (b_i - b_j)$$

(see [1], p. 186) which equals zero only if some pair of the roots are equal. Consequently we have the

THEOREM. *The roots of an n-th degree complex polynomial are analytic functions of the coefficients in the region where $P(z) = 0$, while $P'(z) \neq 0$ for some z.*

3. Multiple roots. Let us next turn to the situation in which the roots are not all distinct. Suppose in fact that the distinct roots are $b_1, \cdots, b_k$ and that these roots occur with multiplicities $p_1, \cdots, p_k$ respectively. (1) becomes

$$\sum_{i=1}^{k} p_i b_i^j = Q_j(a_1, \cdots, a_n).$$

Defining

$$F_j(b_1, \cdots, b_k; a_1, \cdots, a_n) = \sum_{i=1}^{k} p_i b_i^j - Q_j(a_1, \cdots, a_n)$$

for $j=1, \cdots, n$ we see that F_j is analytic everywhere; also the Jacobian is given by

$$k! p_1 \cdots p_k \prod_{1 \leq i < j \leq k} (b_i - b_j).$$

This expression equals zero only if some pair of the b_i are equal. We now have the

THEOREM. *The distinct roots of an n-th degree complex polynomial are analytic functions of the coefficients in the region where the roots retain their various multiplicities.*

We must be careful and note that in this theorem region refers to a region in a subspace of C^n, for since the roots are constrained in their multiplicities, the coefficients $a_1, \cdots, a_n$ are, in fact, constrained to lie in a subspace of C^n.

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1. R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York, 1960.
2. S. Bochner and W. T. Martin, Several Complex Variables, Princeton University Press, Princeton, 1948.
3. W. S. Burnside, A. W. Panton, M. W. J. Fry, The Theory of Equations, Vol. II, Longmans Green, London, 1935.
4. T. Coolidge, Ann. of Math., 9 (1907–8) 116–8.

CURVATURE BY NORMAL LINE CONVERGENCE

JEROME A. SEINER, Pittsburgh Plate Glass Company and Carnegie Institute of Technology

Conventional nonvector procedures for determining the center of curvature lead to an ambiguous solution in which the ambiguity is resolved by an investigation of concavity. It is known that the center of curvature can be found using the normal line. The solution using normals is unambiguous.

Conventional procedures involve simultaneous solution of the systems (1) and (2) and the normal line equation (3)

$$[1 + (f'(a))^2]^{3/2}/f''(a) = R \tag{1}$$

$$(x - a)^2 + (y - f(a))^2 = R^2 \tag{2}$$

$$-1/f'(a) = (y - f(a))/(x - a). \tag{3}$$

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These lead to the ambiguous solution form:

$$x = a \pm f'(a)[(f'(a))^2 + 1]/f''(a) \tag{4}$$

$$y = (a - x)/f'(a) + f(a). \tag{5}$$

Using the normal line we solve equations (3) and (6) simultaneously

$$-1/f'(a + \Delta x) = [y - f(a + \Delta x)]/[x - (a + \Delta x)]. \tag{6}$$

This leads to the normal line intercept function

$$(a - x)[f'(a + \Delta x) - f'(a)] - \Delta x f'(a) = f'(a)f'(a + \Delta x)[f(a + \Delta x) - f(a)]. \tag{7}$$

Dividing both sides by Δx and taking the limit as $\Delta x \to 0$ leads to:

$$x = a - f'(a)[(f'(a))^2 + 1]f''(a) \tag{8}$$

which is the unambiguous form of (4). This combined again with (3) leads to (5).

One application of (8) is to use it as a simple proof of the second derivative test for maxima-minima. If we assign coordinates Yc and Xc to the center of curvature of point $(a, f(a))$, it is intuitively obvious that if Yc is larger than $f(a)$, then $f(a)$ is a relative minima. ($f'(a)=0$).

By combining (5) and (8) one gets then

$$y_c - f(a) = [(f'(a))^2 + 1]/f''(a). \tag{9}$$

Since in this case $f'(a)=0$, this reduces to

$$y_c - f(a) = 1/f''(a). \tag{10}$$

Equation (10) can serve two useful purposes. First, it demonstrates that the sign of $f''(a)$ dictates whether $f(a)$ is a relative maxima or minima. Second, it indicates that the radius of curvature at this point is equal to

$$|1/f''(a)|.$$

The principle of limit normal line convergence can be applied in a similar manner to inflection point analysis to indicate the convergence of (3) with (6) on one side of the curve and with (9) on the other side. Similarly, through rotation of axis, it can be shown that the reciprocal of the second derivative of the rotated function again yields the radius of curvature at critical points relative to the rotated coordinate system.

THE FUNCTION GAME

SCOTT GUTHERY, Miami University

A technique to differentiate functions of a moderately involved nature commonly introduced in freshman calculus is to consider the given function as the composition of several simpler functions and to perform the differentiation using the chain rule. Frequently, however, the student is not sufficiently familiar

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THE FUNCTION GAME

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A technique to differentiate functions of a moderately involved nature commonly introduced in freshman calculus is to consider the given function as the composition of several simpler functions and to perform the differentiation using the chain rule. Frequently, however, the student is not sufficiently familiar with the composition of functions and the associated domain-range considerations to realize the full power of this method. The function game may serve as an educational aid to the teaching of this subject in addition to raising some interesting problems of its own. As an aid, it is an attempt to start the student on familiar (and almost recreational) mathematical grounds and to lead him to the understanding of functional composition through such light-hearted considerations as winning, losing, and strategy.

The kind of functions we shall want to deal with are those loosely referred to as elementary functions. We shall take as an elementary function one that is single- and real-valued and whose domain is a subset of the real numbers. Further, we shall limit ourselves to those function types that are normally encountered in introductory calculus such as polynomial, exponential, and trigonometric functions.

The description of a function game consists of (1) providing each of n players with a set of elementary functions and (2) specifying two real numbers, a and c, called the start and end number respectively. The players move in turn, player 1 up to player n and back to player 1, etc., unless some player has been dropped (as specified below) in which case he is skipped. Player 1 begins a play of the game by evaluating one of his functions at the start number a to obtain, say, b_1. Then player 2 evaluates one of his functions at b_1 to obtain b_2. And so on. The object of the game is to obtain a b_i greater than or equal to the end number c. The player who does this first is proclaimed the winner. If, at the time a player must move, all of his functions are undefined at the b_i, then he is dropped from the game and play continues without him. Finally, if the number of players is reduced to one, then he is the winner. The following examples and the accompanying analysis may help to explain the particulars of the function game. For simplicity, all examples deal with 2-player games.

Suppose player 1's functions are

$$f_{11}(x) = x\sqrt{x}$$
$$f_{12}(x) = \ln x;$$

player 2's functions are

$$f_{21}(x) = 2x$$
$$f_{22}(x) = x - 3;$$

and $a=16$, $c=100$. Player 2 can play so as to win on his first move. For, if player 1 uses f_{11} on his first move, he obtains $b_1=f_{11}(16)=64$ to which player 2 can apply f_{21} to set $b_2=f_{21}f_{11}(16)=f_{21}(64)=128$ and, hence, win the play. If, on the other hand, player 1 uses f_{12} on his first move, he obtains $b_1=f_{12}(16)\doteq 2.78$ to which player 2 can apply f_{22} to obtain $b_2=f_{22}f_{12}(16)\doteq f_{22}(2.78)=-.22$. Since none of player 1's functions are defined for $x<0$, he must drop from the game and player 2 wins by being the only remaining player.

Our second example is a bit more involved. Suppose player 1's functions are

$$f_{11}(x) = 25\sqrt{\{10 - x\}}$$
$$f_{12}(x) = x - 10;$$

player 2's functions are

$$f_{21}(x) = 100\sqrt{x}$$
$$f_{22}(x) = x + 2;$$

and $a=10$, $c=100$. In this game, player 1 has a strategy by which he can assure himself of a win the third time he moves. This strategy is to use f_{12} the first two times he moves and f_{11} the third. If player 1 employs this strategy, there are two possible plays of the game depending upon what player 2 does on his first move. These are

$$10 \to 0 \to 0 \to -10 \to -8 \to 75\sqrt{2},$$

corresponding to the functional composition

$$f_{11}f_{22}f_{12}f_{21}f_{12}(10),$$

and

$$10 \to 0 \to 2 \to -8 \to -6 \to 100,$$

corresponding to the functional composition

$$f_{11}f_{22}f_{12}f_{22}f_{12}(10).$$

Note that player 1 plays to keep player 2 from using f_{21} while playing toward the use of the only function that will win the game for him, f_{11}.

Our final example is one for which player 1 has a strategy that assures him of a win the second time he moves. It is left for solution by the reader. Player 1's functions are

$$f_{11}(x) = 5\sqrt{\{10 - x\}}$$
$$f_{12}(x) = x - 2$$
$$f_{13}(x) = 15\sqrt{\{x - 10\}};$$

player 2's functions are

$$f_{21}(x) = \ln x$$
$$f_{22}(x) = 20\sqrt{\{x - 2\}}$$
$$f_{23}(x) = \sqrt{x};$$

and $a=4$, $c=15$.

Evidently, as the number of players and the number of functions held by the players increases, the game becomes more difficult to analyze. It is also possible that a situation develops in which the players may play *ad infinitum* without reaching or exceeding the end number. In this case, one would want to define a draw, perhaps in terms of a decreasing or repeated sequence of b_i's.

We believe that the function game is capable of providing both valuable practice in functional analysis for the student and some challenging problems for his professor. Certainly the subject of functional composition is one of the most open in mathematics today and perhaps one of the last ones to which a beginner or hobbyist is still capable of making a significant contribution.

GRAPHICAL SOLUTION OF DIFFICULT CROSSING PUZZLES

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1. Introduction. In this article we present a graphical method for solving "difficult crossing" puzzles such as the cannibals and missionaries puzzle or the puzzle of the jealous husbands. The method is extremely simple and makes the solution of many such puzzles easy and quick. It also makes the connection between the two aforementioned puzzles easily apparent.

The basic idea of the method is to regard the players in the melodrama as forming a "system" which can be in a number of "states." A representation of these states and the possible transitions between them is then given by a graph. That is, a graph in the sense of graph theory. See, for example, reference [3]. The idea of using a graph-theoretical approach to the analysis of games and puzzles is not new, of course—see [3]—and has in fact been applied specifically to difficult crossing puzzles by B. Schwartz, [4]. However, Schwartz placed his emphasis on solving the puzzles by matrix operations, and his paper does not as a matter of fact contain any graphs. Apparently the graphical solution to be presented here has not previously been published.

2. The state diagram. Let us first recall the statement of the cannibals and missionaries puzzle, as given by Schwartz.

"A group consisting of three cannibals and three missionaries seeks to cross a river. A boat is available which will hold up to two people. If the missionaries on either side of the river are outnumbered at any time by the cannibals on that side, even momentarily, the cannibals will do away with the unfortunate, outnumbered missionaries. What schedule of crossings can be devised to permit the entire party to cross safely?"

Following Schwartz, we let

$$m = \text{number of missionaries on the first bank},$$

$$c = \text{number of cannibals on the first bank}.$$

Then the pair (c, m) denotes the *state* of the system at any time that the boat is not in midstream. It is not necessary also to give the number of missionaries and cannibals on the second bank since the total number of each must always be three (assuming that the number of missionaries does not suffer an unfortunate decline). Since $0 \leqq m \leqq 3$, $0 \leqq c \leqq 3$, there are sixteen possible pairs, but some of these must be excluded. For example, $m=1$, $c=3$ is excluded. So is $m=2$, $c=1$, since this corresponds to one missionary and two cannibals on the far bank of the river. In fact, the allowable states (c, m) must satisfy these restrictions:

(a) $0 \leqq c \leqq 3$

(b) $0 \leqq m \leqq 3$

(c) $c = m$ or $m = 0$ or $m = 3$.

We now graph the allowable states (c, m) on a rectangular coordinate system, as in Fig. 1. There are ten states, lying in the shape of a letter Z. We call this figure the *state diagram* of the system.

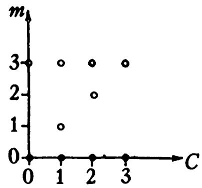

FIG. 1.

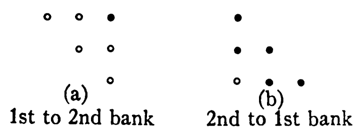

FIG. 2. Reachable points

3. Transitions between states. Transitions between states can be shown geometrically as directed lines or arcs joining the points of the state diagram. In this way, a *directed graph* is obtained, the points or *vertices* corresponding to states and the *edges* corresponding to crossings of the river. Which vertices will thus be joined by edges? We observe that a passage of the boat from the first to the second bank reduces the values of c and m since it removes people from the first bank. The total decrease in c and m is either one or two since the capacity of the boat is two. Thus the only points of the graph which are reachable from a given point on such a passage lie in a right triangle with the given point at the upper right vertex, as in Fig. 2a. Of course, some of these reachable points may not be allowable. Similarly passage of the boat from the second to the first bank of the river corresponds to increasing c or m. The points reachable from a given point in such a transition are shown in Fig. 2b.

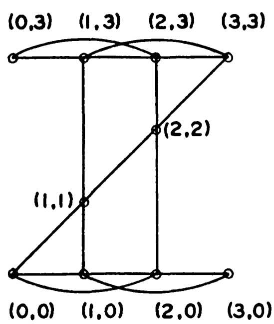

FIG. 3.

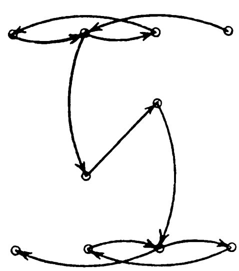

FIG. 4.

Keeping these rules in mind, we can construct the graph of the puzzle, shown in Fig. 3. Each edge can be traversed in either direction, corresponding to the fact that all ferryings are reversible (by turning the boat around and returning its passengers to their starting point). It should be kept in mind that passages from the first to the second bank must alternate with passages from the second to the first bank. This means that on the graph edges leading down or to the left (to reachable points as in Fig. 2a) must alternate with edges leading up or to the right (to reachable points as in Fig. 2b). Thus, (3, 3) to (2, 2) to (1, 1) to (0,0) is not a permissible solution of the puzzle.

A method for finding a solution to the puzzle can now be described in the following terms. Starting at the upper right corner of the graph, make a sequence of transitions to allowable points in the reachable triangles, moving alternately down or to the left and up or to the right, until the lower left corner is reached. After a little experimentation, the solution shown in Fig. 4 is obtained.

4. Existence, uniqueness, minimization. It is natural to ask whether the solution shown in Fig. 4 is the only solution to the puzzle. More generally, if one starts with some other numbers of cannibals and missionaries, is the puzzle solvable, and if so which solution (if there are several) requires the fewest crossings of the river?

The graphical method is of assistance in answering some of these questions. Let us classify the allowable states into three kinds:

(T) those along the top ($m=3$),

(B) those along the bottom ($m=0$),

(D) those along the diagonal ($0<m=c<3$).

If the boat holds only two people, it is impossible to go directly from a state of type T to one of type B. Hence any solution path must include a type D state. Moreover, if we move from (2, 3) or (3, 3) to (2, 2), then at the next step we must go to (2, 3) or (3, 3), and nothing is achieved. Therefore when the solution path leaves the T states, it must go to (1, 1). The only way to do this is from (1, 3). The only way to reach B states is then to go to (2, 2) and (2, 0). Thus, the solution path must contain the sequence

$$\cdots \rightarrow (1, 3) \rightarrow (1, 1) \rightarrow (2, 2) \rightarrow (2, 0) \rightarrow \cdots$$

Finally, (1, 3) can be reached from (3, 3) by

$$(3, 3) \rightarrow (1, 3) \rightarrow (2, 3) \rightarrow (0, 3) \rightarrow (1, 3)$$

or

$$(3, 3) \rightarrow (2, 2) \rightarrow (2, 3) \rightarrow (0, 3) \rightarrow (1, 3)$$

and similarly (0, 0) can be reached from (2,0) in two ways. Hence there are exactly four possible distinct solutions of the puzzle. All four require eleven crossings of the river (five and a half round trips).

In the same way, reasoning from the graphical representation makes it easy to answer questions of the existence and uniqueness of solutions and choice of the solution requiring a minimal number of crossings for certain more general puzzles. For example, four cannibals and four missionaries cannot be taken safely across a river with a boat holding only two people. To see this, we refer to the state diagram in Fig. 5. In order to leave the T states without immediately returning, one must go from (2, 4) to (2, 2). The next step must be to (3, 3), or back to (2, 4). From (3, 3), there is no way to get to the bottom states, nor to (1, 1), and the puzzle has no solution.

If the boat will hold three people instead of two, the reachable states are as indicated in Fig. 6. (We require that the boat not contain more cannibals than missionaries.) It is easy to see that up to five cannibals and five missionaries can now safely cross the river, but not six cannibals and six missionaries.

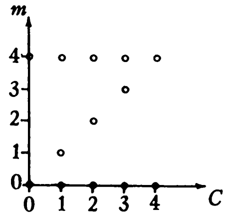

FIG. 5.

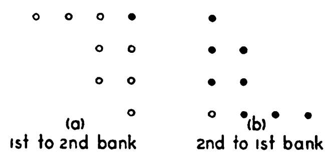

FIG. 6.

On the other hand, if the boat will hold four or more people, then any number of missionaries and cannibals can be transported across the river, for it is then possible to move down along the diagonal of the state diagram, as shown in Fig. 7 for the case of six cannibals, six missionaries, and a boat holding four. The diagonal solution is not always the one requiring the fewest crossings, however. For example, if there are six missionaries and six cannibals and the boat holds five, the solution which sticks to the diagonal is still the one in Fig. 7, but the solution in Fig. 8 requires only seven crossings. However, one can show that if the boat holds an even number, B, of people, and $4 \leqq B < M$ where M is the number of missionaries (or cannibals), then no path can reach (0, 0) in

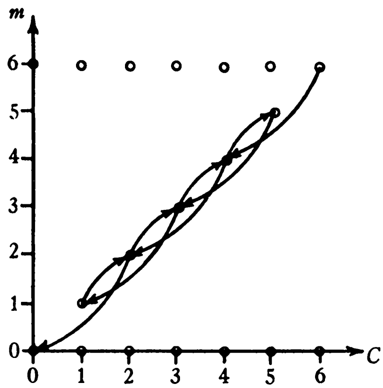

FIG. 7.

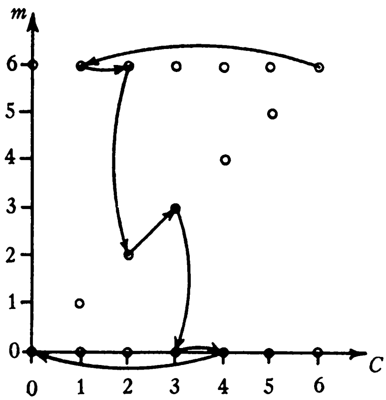

FIG. 8.

fewer steps than the one down the diagonal. To see this, we first note that the path must pass through a diagonal state and the best way to leave the top states (other than to follow the diagonal) is to send B cannibals across on the first step, return one cannibal, and then send $B-1$ missionaries, as shown in Fig. 9a. Thus a net of $B-1$ missionaries and $B-1$ cannibals is transported in three crossings. On the other hand, the same result can be achieved by following the diagonal as in Fig. 9b. Similarly one can show that no advantage can be gained by leaving the diagonal to go to the bottom states when this becomes possible. For if this is done, at most $2B-1$ persons can be transported in 3 steps versus $2B-2$ for the diagonal route, and since it is necessary to transport an even number of persons the difference cannot make it possible to reach

(0, 0) in fewer steps (see Fig. 10, where two possible cases are depicted). We leave it to the reader to analyze the case when B is odd.

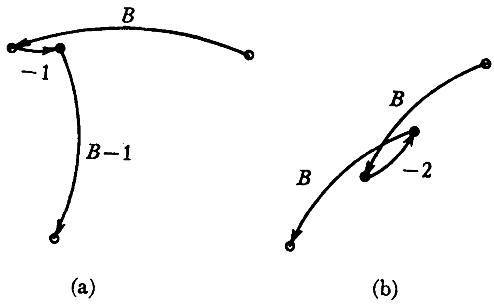

FIG. 9. (Numbers beside arcs indicate numbers of persons transported from 1st to 2nd bank.)

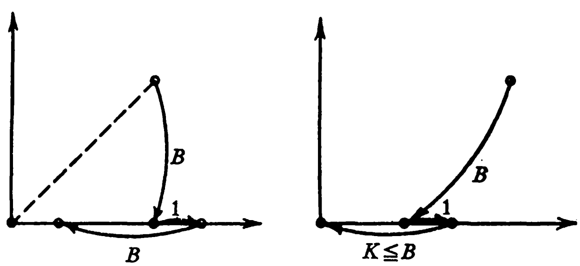

FIG. 10.

5. More general puzzles. The graphical technique makes it easy to analyze some simple variants of the basic puzzle. For example, it may be specified that the boat not only has a maximum capacity but also a minimum capacity (greater than one). In fact, arbitrary constraints can be imposed, for this simply means a change in the set of allowable states to be shown on the state diagram. This set of allowable states can even be a function of the number of crossings, the last state, or the sequence of states which has been used. In complicated cases of this sort, it may be necessary to use a computer to carry out the solution.

If the number of types of individuals is increased, the puzzle becomes more complicated. For example, Schwartz uses his analytic method to discuss a puzzle in which there are M missionaries, all of whom can row, R cannibals who can row, and C cannibals who cannot row. The state of the system is then described by an ordered triple of numbers (c, r, m). The graph of this puzzle can be shown on the plane in a convenient way by placing all points with the same value of m on a horizontal line. The complete graph for the case $M=3$, $R=1$, $C=2$ is shown in Fig. 11; from it the solution $(2, 1, 3)\to(1, 0, 3)\to(1, 1, 3)\to(0, 0, 3)\to(0, 1, 3)\to(0, 1, 1)\to(1, 1, 2)\to(1, 0, 1)\to(2, 0, 2)\to(2, 0, 0)\to(2, 1, 0)\to(1, 0, 0)\to(1, 1, 0)\to(0, 0, 0)$ can be obtained. In more general situations, with many classes of individuals, it may be necessary to use computer solutions based on the analytical method of Schwartz or the dynamic programming method of Bellman, [1].

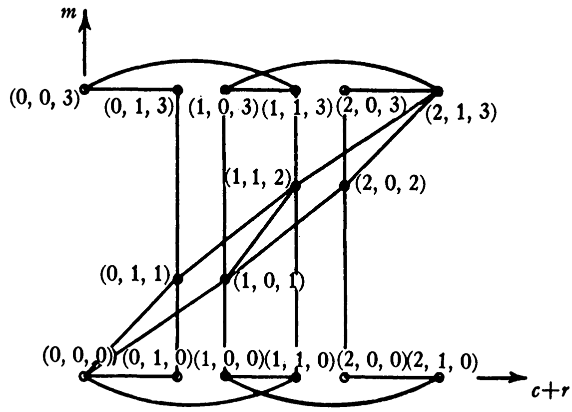

FIG. 11. State (c, r, m)

6. Puzzle of the jealous husbands. We shall close by demonstrating the relation of the puzzle of the jealous husbands to the preceding discussion. This puzzle may be stated as follows.

"Three jealous husbands and their wives must cross a river in a boat that holds only two persons. How can this be done so that a wife is never left in the company of either or both of the other women's husbands unless her own husband is present?"

This puzzle is an ancient one whose origin is apparently unknown. The puzzle was known to the famous Italian mathematician Tartaglia (born 1510, died 1557), who gave an incorrect solution of the puzzle with four families. (See Lucas, [2], for additional historical information.) The cannibals and missionaries puzzle is a later version.

In order to obtain a graphical interpretation of this new puzzle, we shall define the states to be ordered pairs (w, h), where h is the number of husbands and w is the number of wives on the first bank of the river. At first sight, this seems to be an unsatisfactory definition because seemingly different situations will be put together into a single state. For example, if we refer to the three husbands as A, B, and C and their respective wives as a, b, and c, then the situation in which the persons on the first bank are A and a and those situations in which they are A and b or A and c all have $h=w=1$. Thus, all are classed together as the state (1, 1), although the latter two are not allowable. However, since we intend to show only allowable states on our state diagram this is not really an objection. It will still be true that the state (1, 1) can be realized by having A and a, B and b, or C and c on the left bank, but we have no need to distinguish among these cases.

Now that the states of the system have been defined, we can draw the state diagram. The result is exactly the same as Fig. 1, except that the labels on the axes must be changed from c, m, to w, h. It is to be understood that the diagonal states are allowable only if the husbands and wives are matched. Thus (2, 2) can be achieved by A, B, a, b or A, C, a, c or B, C, b, c but not by A, B, a, c, etc. It is now clear that solutions of the cannibals and missionaries puzzle such as in Fig. 4 provide the only possible solutions of the husbands puzzle. To see that Fig. 4, for example, actually yields a solution of the husbands puzzle, it is only necessary to check that the transitions $(1, 3)\rightarrow(1, 1)\rightarrow(2, 2)$ can be made without violating the requirement that the husbands and wives be matched, as is indeed the case.

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Editorial Comment on r! Several readers have written in to point out that the result

$$r! = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n - i)^r$$

conjectured by Tepper [5] and proved by Long [3] was known as early as the time of L. Euler; see [1, p. 62]. Essentially a well-known result in the calculus of finite differences, it can readily be deduced from the fact that $\Delta^r f(x) = a_r r!$ for any polynomial $f(x)$ of degree r with leading coefficient a_r and that

$$\Delta^r f(x) = \sum_{i=0}^{r} (-1)^r \binom{r}{i} f(x + r - i).$$

One has only to consider the special case $f(x) = (x - r)^r$. See [4; p. 10 and problem 4, p. 19]. For two additional proofs, see also [2].

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MORE ON FINITE SUBSETS AND SIMPLE CLOSED POLYGONAL PATHS

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In a recent paper [1] in this MAGAZINE, it was shown that given any finite subset F of the plane R^2 such that all the points of F are not collinear, there is at least one simple closed polygonal path (s.c.p.p.) which has F as its set of vertices. This paper extends this result to R^m, $m>2$.

THEOREM. *Given any finite subset F of R^m, $m \geqq 2$, such that all the points of F are not collinear, there is at least one s.c.p.p. for which F is the set of vertices.*

Proof. This theorem has already been proved for $m=2$. Assume the theorem is true for $m-1 \geqq 2$ and let F be any finite subset of R^m, $m \geqq 3$, such that all the points of F are not collinear. If F is contained in any plane, the theorem holds; hence suppose also that F is not contained in any plane.

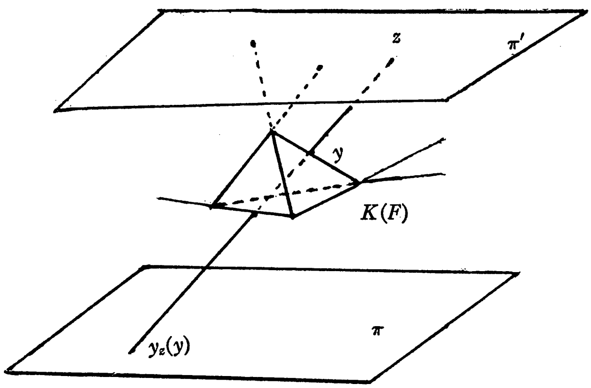

FIG. 1.

Since F is finite we can find parallel hyperplanes π and π' such that F is contained "between" π and π'. (Figure 1 gives the picture in going from R^2 to R^3.) Let T be the set of points determined by the intersection with π' of all lines in R^m which contain two or more points of F. Let z be any point in $\pi'-T$ and $K(F)$ be the set of all segments joining pairs of points in F. The region bounded by π and π' is convex since it is the intersection of two convex regions. Therefore $K(F)$ also lies in this region. Let j_z be the projection of $K(F)$ onto π from z, i.e., if $y \in K(F)$, then $j_z(y)$ is the intersection with π of the line determined by y and z. Since the points of F are not all coplanar, the points of $j_z(F)$ are not all collinear, for if they were then F would be in the plane determined by $j_z(F)$ and z. Since $z \in \pi'-T$, $j_z | F$ is $1-1$, and the image of $K(F)$ is the set of all segments joining pairs of points in $j_z(F)$. By the induction hypothesis, $j_z(K(F))$ contains a s.c.p.p. C for which $j_z(F)$ is the set of vertices. Then $j_z^{-1}(C)$ is a s.c.p.p. for which F is the set of vertices.

If x and y are distinct points of the m-sphere S^m, a segment joining x and y

could be defined as any arc of a great circle which has x and y as end points. A polygonal path in S^m would then be a path composed of such segments.

THEOREM. *Given any finite subset F of two or more points of S^2, there is at least one s.c.p.p. which has F as its set of vertices.* (The author is indebted to Prof. G. Itzkowitz for a suggestion which led immediately to a proof.)

Proof. Let $F=\{x_0, \cdots, x_m\}$. If F is contained in a great circle, then the great circle is a s.c.p.p. having F as its set of vertices. Suppose F is not contained in a great circle. Since F is finite, we can find some point $w\in S^2$ such that neither w, nor w', the point antipodal to w, are in F or in any great circle containing any two nonantipodal points of F. Let h_i be the segment joining w and w' which contains x_i, $i=0, \cdots, m$. Let C' be any great circle which contains x_0, but not w. Beginning with h_0 we can order the h_i according to the order we encounter their intersections with C' proceeding around C' is a clockwise direction. We may assume this is the natural numerical order; we shall set $h_{m+1}=h_0$ and $x_{m+1}=x_0$. Then one of the two segments joining x_0 and x_{i+1}, $i=0, \cdots, m$, lies entirely in the wedged shaped area of the sphere of which h_i and h_{i+1} form the boundary; we denote this segment by $\overline{x_ix_{i+1}}$. Then $\overline{x_0x_1}\cup\overline{x_1x_2}\cup\cdots\cup\overline{x_{m-1}x_m}\cup\overline{x_mx_0}$ is a s.c.p.p. of which F is the set of vertices.

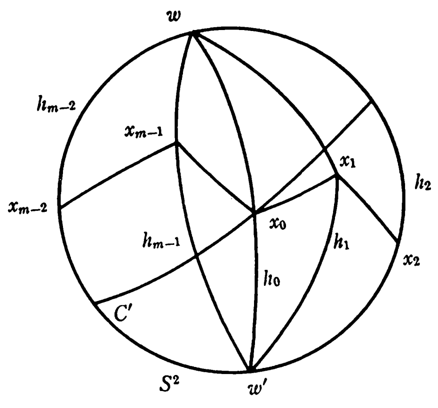

FIG. 2.

The reader will note that a modified form of the above proof can also be used to prove Theorem 1 of [1]: If $F=\{x_0, \cdots, x_m\}$ is a finite subset of the plane such that all the points of F are not collinear, then we can choose w in the convex hull of F and not on any of the lines determined by pairs of points of F, and apply the methods of the above proof in obvious fashion. The resulting s.c.p.p. C we obtain has w as a star point, that is, the segments $\overline{wx_i}$, $i=0, \cdots, m$, intersect C only in x_i. Since it is entirely possible that many s.c.p. paths having F as their set of vertices do not have any star points, the above procedure could not be used to enumerate all s.c.p. paths having F as the set of vertices. The proof in [1] not only shows existence, but points to the possibility of determining the number of s.c.p. paths having F as the set of vertices.

The author notes that the conjecture which closes [1] has been proved in [2].

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2. L. Quintas and F. Supnick, On some properties of shortest Hamiltonian circuits, Amer. Math. Monthly, 72 (1965) 977–980.

ANSWERS

A383. $F_4 < L_3 < F_5 < L_4 < F_6$. The Lucas-Fibonacci recursion guarantees that $F_6 < L_5 < F_7 < \cdots$. By induction, the two sequences remain strictly interleaved. Hence, 1 and 3 are the only numbers common to both sequences.

A384. Extend DL, EM, FN to meet at the circumcenter O. Since

$$OL + OM + ON = R + r,$$

we have

$$3R - (DL + EM + FN) = R + r$$

or

$$2R = DL + EM + FN + r.$$

A385. Let P, Q, R be any three distinct points at infinity. Choose real lines p, q, r to pass through these points, respectively. These lines determine a real triangle ABC. Now P, Q, R divide the sides of triangle ABC in the ratio -1. $(-1)(-1)(-1) = -1$. Hence P, Q, R are collinear by the theorem of Menelaus.

A386. We have

$$p_1 < \frac{p_1 + p_2}{2} < p_2$$

and

$$q = \frac{p_1 + p_2}{2} \cdot$$

Thus, since p_1 and p_2 are successive primes, q is not a prime.

A387. Plant five trees forming the pentagon 1 2 3 4 5. Then plant five at these intersections of pairs of opposite sides, as (1 2) (34), $\cdots$, (5 2) (1 3). Plant five more at the intersections of pairs of diagonals, as (1 3) (2 4), $\cdots$, (5 2) (1 3), and plant the last five at the intersections of side-diagonal pairs, as (1 2) (3 5), $\cdots$, (5 1) (2 4). The first of the four pentagons must be such that none of the 15 pairs of lines are parallel, so that it can not, for example, be regular on any finite form.

(Quickies on page 196)

ANOTHER REMARK CONCERNING THE DEFINITION OF A FIELD

HERBERT E. VAUGHAN, University of Illinois

The title refers to articles by A. H. Lightstone [1] and Joseph J. Malone, Jr. [2] in this MAGAZINE. Lightstone showed by an example that if $+$ and $\cdot$ are binary operations on a set S, $0 \in S$, $1 \in S$, and

(i) $(S, +, 0)$ is an abelian group,
(ii) $(S-\{0\}, \cdot', 1)$ is an abelian group, where $\cdot'$ is $\cdot$ restricted to $S-\{0\}$,
(iii) (a) $0 \neq 1$
(b) $\forall_x \forall_y \forall_z [x\cdot(y+z) = x\cdot y\cdot + x\cdot z]$,

then $(S, +, \cdot, 0, 1)$ need not be a field. Malone showed that Lightstone's example is the only model (up to isomorphism) of (i), (ii), and (iii) (a), (b) which is not a field and for which $\cdot$ is associative. [Of course, by (ii), $\cdot'$ is associative.]

The purpose of this note is to call attention to the rather obvious remark that the models of (i), (ii), and (iii) (a) (b) are fields and the systems which are obtained from fields by redefining multiplication so that multiplication (on the left) by 0 is some given endomorphism of the group $(S, +, 0)$. For example, if S is the set of complex numbers, $+$ is addition of complex numbers, $\cdot$ is defined so that multiplication on the left by a nonzero complex number is complex multiplication, and multiplication on the left by the complex number 0 is conjugation, then $(S, +, \cdot, 0, 1)$ is a model of (i), (ii), and (iii) (a) (b). As other examples, if $(S, +, \times, 0, 1)$ is a field and $\cdot$ is defined so that, for any $b \in S$, $a\cdot b = a\times b$ if $a \neq 0$, and $0\cdot b = u \times b$, for some given $u \in S$, then $(S, +, \cdot, 0, 1)$ is a model of (i), (ii), and (iii) (a) (b).

There are several ways to augment (i), (ii), and (iii) (a) (b) to obtain field postulates. One may, as Lightstone notes, adjoin:

(iii) (c) $\forall_x \forall_y \forall_z \ [(y+z)\cdot x = y\cdot z + z\cdot x]$.

Somewhat more economically, one may adjoin:

(iii) (c′) $\forall_{x\neq 0} 0\cdot x = 0\cdot x + 0\cdot x$

Most directly, one may adjoin:

(iii) (c″) $\forall_{x\neq 0} 0\cdot x = 0$

Malone's result—that it is sufficient to assume that $\cdot$ is associative and S has more than two members—may be refined to show that it is sufficient to postulate:

(iii) (c‴) $\forall_{x\neq 0} \forall_{z\neq 0} z\cdot(0\cdot x) = 0\cdot x$ and S has more than two members

or:

(iii) (c^{iv}) $\forall_{x\neq 0} \exists_{z\notin\{0,1\}} z\cdot(0\cdot x) = 0\cdot x$.

Finally, to restrict models to those obtained from fields by defining multiplication by 0 to be a multiplicative endomorphism of their additive groups, it is sufficient to adjoin the postulate:

(iii) (c^v) $\forall_{x\neq 0}\forall_{y\neq 0}\, y\cdot(0\cdot x)=0\cdot(y\cdot x)$

or even:

$$\forall_{y\neq 0}\, y\cdot(0\cdot 1) = 0\cdot(y\cdot 1).$$

References

1. A. H. Lightstone, A remark concerning the definitions of a field, this MAGAZINE, 37 (1964) 12–13.

2. Joseph J. Malone, Jr., An additional remark concerning the definition of a field, ibid., 38 (1965) 94.

A GEOMETRICAL SOLUTION OF THE THREE FACTORY PROBLEM

W. J. VAN DE LINDT, IBM Systems Research and Development Center, Los Angeles

The problem can be stated as follows:
Given three points, a, b, c. Each one is connected with a fourth point, u. Suppose the length of $ua=r_1$, $ub=r_2$, and $uc=r_3$. Choose the point u so that $C=k_1r_1+k_2r_2+k_3r_3$ is minimized. k_1, k_2, k_3, are arbitrary positive weighing factors. Figure 1 shows a possible situation.

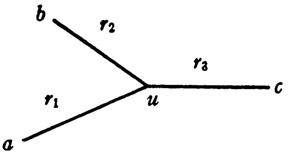

FIG. 1.

This problem was described by Greenberg and Robertello [1] and solved algebraically.

The interpretation is that three factories a, b and c are to be supplied with commodities k_1, k_2 and k_3, respectively from a central warehouse u. The cost, which is proportional to distance and quantity, is to be minimized. The location of u is the variable.

The identical problem occurs when, for instance, oil or gas is transported by pipeline from two locations to a common destination. The two pipelines join in u and continue as one. To be chosen is the location of u. In general the three pipelines have different diameters and consequently, cost differently. Besides the algebraic solution given [1], it is possible to solve this problem by means of an elementary geometric construction with the help of ruler and compass.

Finally, to restrict models to those obtained from fields by defining multiplication by 0 to be a multiplicative endomorphism of their additive groups, it is sufficient to adjoin the postulate:

(iii) (c^v) $\forall_{x\neq 0}\forall_{y\neq 0}y\cdot(0\cdot x)=0\cdot(y\cdot x)$

or even:

$$\forall_{y\neq 0}y\cdot(0\cdot 1) = 0\cdot(y\cdot 1).$$

References

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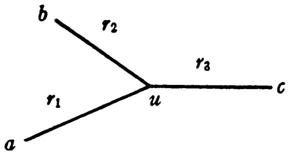

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The identical problem occurs when, for instance, oil or gas is transported by pipeline from two locations to a common destination. The two pipelines join in u and continue as one. To be chosen is the location of u. In general the three pipelines have different diameters and consequently, cost differently. Besides the algebraic solution given [1], it is possible to solve this problem by means of an elementary geometric construction with the help of ruler and compass.

To find this construction, it is useful to calculate the change in total cost C, if, starting in an arbitrary point u, a new point u' is chosen, lying close to the original point u. Let the displacement vector uu' be called $\bar{\Delta}$. (See Fig. 2.)

The new distances r_i' are given by

$$r_i' = \sqrt{\{r^2_i + \Delta^2 - 2r_i\Delta \cos \phi_i\}} \approx r_i - \Delta \cos \phi_i$$

for small enough Δ. Consequently, the change in C, due to the displacement is

$$\Delta C \approx - k_1\Delta \cos \phi_1 - k_2\Delta \cos \phi_2 - k_3\Delta \cos \phi_3.$$

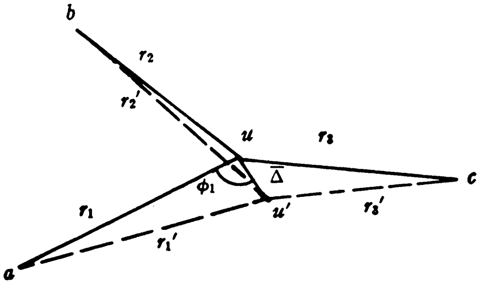

FIG. 2.

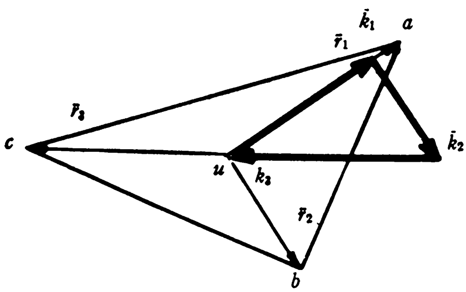

FIG. 3.

If the vectors $\bar{k}_i$ are defined as vectors lying in the direction $\bar{r}_i$ and having a ength k_i, so that

$$\bar{k}_i = \frac{k_i\bar{r}_i}{[r_i]},$$

the expression for ΔC can be written in vector form as

$$\Delta C = - (\bar{k}_1 + \bar{k}_2 + \bar{k}_3)\cdot\bar{\Delta}.$$

Now in the point where C is minimum, this change must be zero for any small vector $\bar{\Delta}$. It follows then from the expression for ΔC that $\bar{k}_1+\bar{k}_2+\bar{k}_3=0$ in the point where C is a minimum.

Solution. If the three vectors $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$, add up to zero, they must form a triangle as shown in Figure 3. This immediately gives the relative directions of the three vectors $\bar{k}$, as well as those of $\bar{r}$. This is shown in the same figure. In the triangles *abu*, *bcu*, the bases as well as the vertices are known. It is known that the point u lies on a circle through a and b in such a way that the arc on which the angle *aub* stands is equal to twice the angle *aub*. This circle can be constructed as is shown in Figure 4. All this is true for *buc* too. The point where the two circles intersect is the point where C is a minimum. Figure 5 gives a complete construction of an example.

Discussion. If the three quantities k_i are of such a magnitude that no triangle can be formed, a minimum does not exist. Also, a minimum must lie inside the triangle. This can be seen by considering a point on the boundary. Going out of the triangle all three r_i increase, thereby increasing C. This imposes a condition for a minimum to exist. From Figure 5 it can be seen that, for instance $\alpha_1 > \beta_1$. This gives

$$\cos \alpha_3 = \frac{k_3^2 - k_1^2 - k_2^2}{2k_1k_2} < \frac{l_1^2 + l_2^2 - l_3^2}{2l_1l_2} = \cos \beta_3$$

and similar expressions for α_2 and α_1. However, it is easier to construct the point u and verify that it falls within the triangle. That it is a minimum and not a maximum can also be checked by calculating some other points.

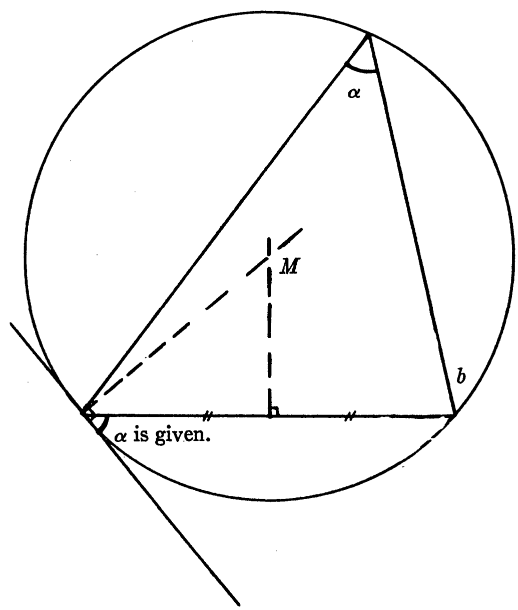

FIG. 4.

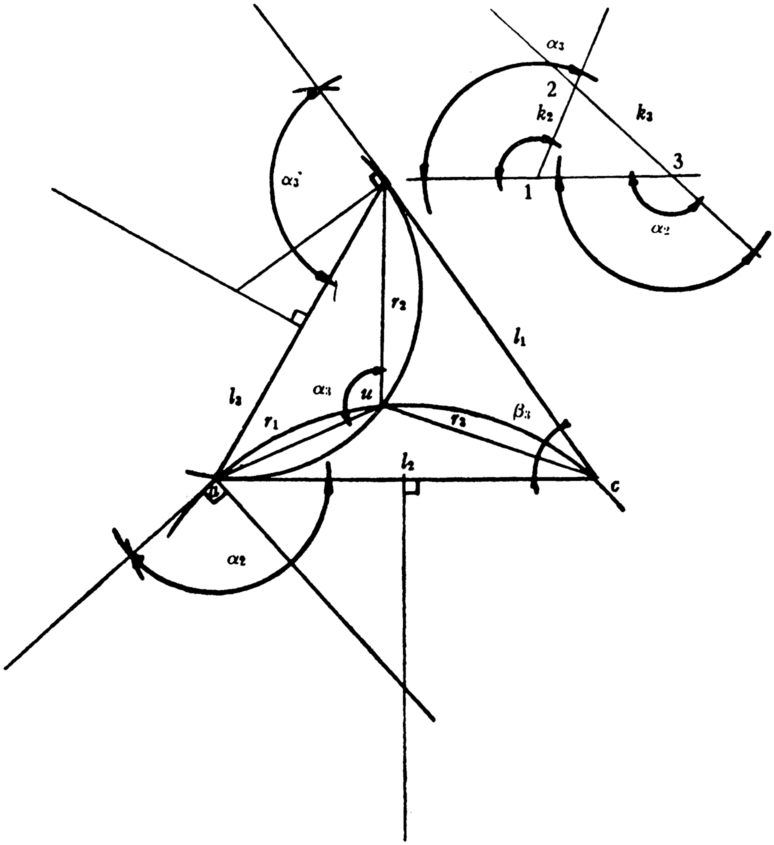

FIG. 5.

Reference

1. I. Greenberg and R. A. Robertello, The three factory problem, this MAGAZINE, 38 (1965) 67–72.

SIMILAR TRIANGLES

J. G. MAULDON, Corpus Christi College, Oxford

The object of this paper is to demonstrate a simple but powerful technique for discovering and proving theorems involving similar triangles in a plane. In particular we obtain a number of new results about equilateral triangles, including two distinct generalizations of a result in [1]. We also initiate a survey of plane systems of triangles whose incidence relations are such that, if all but one are directly similar (e.g. equilateral), then they all are.

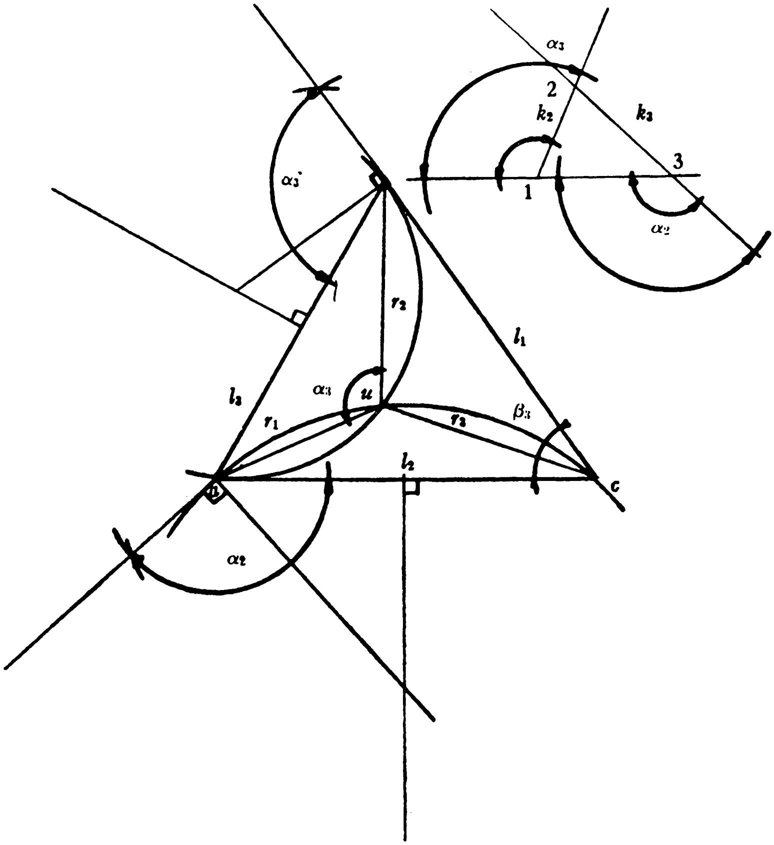

FIG. 5.

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1. Hyper-oriented triangles. We shall use the unqualified word *triangle* to denote an ordered set of three points (its *vertices*) and occasionally, when we wish to emphasize the fact that the vertices are taken in a definite order, we shall refer to a triangle as a *hyper-oriented triangle.* In a similar way, an *unoriented triangle* may be regarded as an unordered set of three points and an *oriented triangle* as a set of three points whose cyclic order is specified. Thus to any non-degenerate unoriented triangle correspond two distinct oriented triangles and six distinct triangles, all with the same set of three vertices.

2. Directly similar triangles. If ABC, XYZ are two triangles in the complex plane whose vertices are the complex numbers a, b, c, x, y, z, then the relation

$$\begin{vmatrix} a & b & c \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{1}$$

holds when the triangles are identical and is preserved under translation, rotation, and dilation of each triangle separately. Hence, as is easily proved directly, (1) is a necessary and sufficient condition for the direct similarity of the triangles ABC, XYZ. The condition for inversely similar triangles is obtained by replacing either a, b, c or x, y, z by their complex conjugates. Notice that direct similarity is not a transitive relation, since we admit the degenerate triangle XYZ with $x=y=z$.

In particular, taking $x=1$, $y=\omega$, $z=\omega^2$, where $2\omega=-1+i\sqrt{3}$, we find that the triangle ABC is equilateral if and only if

$$\text{either } a+\omega b+\omega^2 c=0 \qquad \text{or } a+\omega^2 b+\omega c=0. \tag{2}$$

In the former case we shall say that the triangle ABC is *directly equilateral* and in the latter case *inversely equilateral.* It should be observed that these descriptions are applicable also to oriented triangles, since, if one of the three corresponding hyper-oriented triangles is directly [inversely] equilateral, then they all are. We may therefore speak unambiguously of the vertex of the inversely equilateral triangle erected on a given directed segment.

As an example of the technique we first give a short proof of "Napoleon's Theorem" (cf. [1]).

THEOREM (Napoleon). *The vertices P_i of inversely equilateral triangles erected on the middle thirds of the sides of an oriented triangle $\{A_i\}$ $(A_{i+3}=A_i)$ form a directly equilateral triangle.*

Proof. We shall throughout identify points in the complex plane with complex numbers denoted by the corresponding lower case letters. Observing here that the triangle $P_iA_{i+1}A_{i+2}$ is directly similar to the triangle $0\omega 1$ we have, from (1),

$$(\omega-1)p_i+a_{i+1}-\omega a_{i+2}=0 \qquad (i=1, 2, 3), \tag{3}$$

from which the required result $p_1+\omega p_2+\omega^2 p_3=0$ follows immediately.

A slightly more difficult theorem to prove is

THEOREM 1. *If three coplanar equilateral triangles are appended one to each vertex of a fourth equilateral triangle, then the midpoints of the joins of consecutive free vertices are the vertices of an equilateral triangle.*

This theorem is illustrated in Figure 1, from which it will be seen that "*consecutive free vertices*" are connected by a sequence of three oriented sides of different triangles, the four original triangles being all similarly oriented.

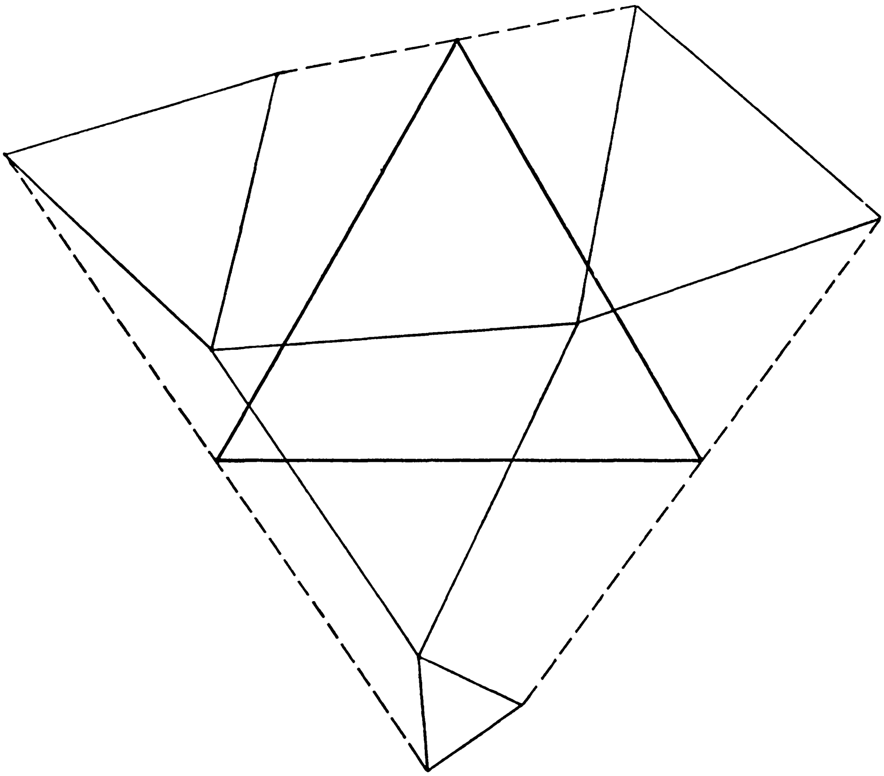

FIG. 1.

Proof of Theorem 1. If the vertices of the "fourth triangle" are taken to be $1, \omega, \omega^2$, then the "free vertices" may be taken to be $1+x$, $1-\omega^2x$, $\omega+y$, $\omega-\omega^2y$, ω^2+z, $\omega^2-\omega^2z$, the "midpoints" P_1, P_2, P_3 have coordinates $\frac{1}{2}(1-\omega^2x+\omega+y)$, $\frac{1}{2}(\omega-\omega^2y+\omega^2+z)$, $\frac{1}{2}(\omega^2-\omega^2z+1+x)$, and the required relation $p_1+\omega p_2+\omega^2p_3=0$ is immediate.

Theorem 1 is a special case of Theorem 8 below. Further striking theorems on equilateral triangles will be found as corollaries of Theorems 3, 4 and 8 (see also Theorem 7).

THEOREM 2. *If directly similar triangles are erected on the sides AB, CB, CD, AD of a plane quadrilateral $ABCD$, then their vertices form a parallelogram.*

Proof. Denoting the four vertices by P, Q, R, S, we have, from (1), $p=\lambda a$

$+\mu b$, $q=\lambda c+\mu b$, $r=\lambda c+\mu d$, $s=\lambda a+\mu d$, where $\lambda=(b-p)/(b-a)$ and $\mu=(p-a)/(b-a)$. Hence $\frac{1}{2}(p+r)=\frac{1}{2}(q+s)$ so that, as required, the midpoints of the diagonals of $PQRS$ coincide.

A particular case of Theorem 2 is the

COROLLARY. *If the coplanar triangles PAB, QCB, RCD, SAD, and SPQ are all directly similar, then so is the triangle QRS.*

THEOREM 3. *If $A_1A_2A_3$ and $B_1B_2B_3$ are distinct coplanar directly similar triangles, and if $\{A_rB_rC_r: r=1, 2, 3\}$ is a set of three coplanar directly similar triangles, then the triangle $C_1C_2C_3$ is directly similar to $A_1A_2A_3$ and $B_1B_2B_3$.*

Proof. We may assume that $a_1 \neq b_1$ and write $\lambda=(c_1-b_1)/(a_1-b_1)$, $\mu=(a_1-c_1)/(a_1-b_1)$. Then $c_r=\lambda a_r+\mu b_r$ $(r=1, 2, 3)$, and the required conclusion is immediate.

COROLLARY. *The midpoints of the joins of corresponding vertices of a pair of coplanar directly equilateral triangles are the vertices of a directly equilateral triangle.*

3. Configurations. Remember that a *triangle* is an ordered triad of points (its *vertices*). If the ith vertex of one triangle coincides with the jth vertex of another, then this fact is called an *incidence relation* between the triangles.

We define a *configuration* to be a finite set of incidence relations between triangles. Such a set of incidence relations is most conveniently denoted by listing the corresponding triangles, using the same symbol for vertices identified by the relations. For example, $\{ABC, BCD\}$ denotes a configuration of two incidence relations and $\{ABC, ABD, ACD\}$ denotes a configuration of six incidence relations, of which only five are independent. We include in the symbol for a configuration only those triangles which are actually subject to at least one of the incidence relations, and these triangles are called the *component triangles* of the configuration.

4. Similarity systems. A *similarity system* is a configuration such that, if its component triangles are coplanar and if all but one of them are directly similar (e.g. directly equilateral), then they are all directly similar. Since we include the possibility that the component triangles may be degenerate, it follows that contrary to first impressions given by the corollary to Theorem 2, the configuration $\{PAB, QCB, RCD, SAD, SPQ, QRS\}$ is *not* a similarity system since, if B and P coincide and D, Q, R, S coincide at a different point, all except the second of the six component triangles are directly similar. Similarly (cf. Theorem 3) the configuration $\{A_1A_2A_3, B_1B_2B_3, C_1C_2C_3, A_rB_rC_r: r=1, 2, 3\}$ is *not* a similarity system since, if all the points except A_3 coincide, then all the component triangles except the last (and also all except the first) are directly similar.

A well-known similarity system is the configuration $\{ABC, ARQ, RBP, QPC, PQR\}$ and a more unfamiliar example (proof left to the reader) is the configuration $\{OAB, COB, CDO, ODE, FOE, FAO\}$. In the next section we introduce a natural and symmetric configuration of eight component triangles which turns out to be a similarity system.

5. A particular similarity system. Consider a rectangular parallelepiped such that the three sets of parallel edges are all of different length. Then the eight triads of incident edges, ordered according to length, may be denoted by

$$\{A_1B_1C_1,\ A_2B_2C_2,\ A_3B_3C_3,\ A_4B_4C_4,\ A_1B_4C_3,\ A_2B_3C_4,\ A_3B_2C_1,\ A_4B_1C_2\}. \tag{4}$$

If we now regard these eight ordered triads as triangles, (4) denotes a configuration, and we have

THEOREM 4. *The configuration* (4) *is a similarity system.*

This theorem is illustrated in Figure 4. The reason for postponing the illustration is that Theorem 4 is one of two completely different generalizations of a recently published theorem, and we wish to prove the other generalization and then compare the two.

Proof of Theorem 4. The condition that the triangle $A_1B_1C_1$ should be directly similar to (say) the triangle XYZ is $(y-z)a_1+(z-x)b_1+(x-y)c_1=0$. Thus the sum of the conditions for the first four triangles of the configuration (4) is the same as the sum of the conditions for the remaining four so that, as required, any seven of the conditions imply the eighth.

If we regard $A_1B_1C_2A_2B_3C_3$ as an oriented hexagon and take the particular case of Theorem 4 in which the eight triangles are all inversely equilateral, we obtain the

COROLLARY. *If inversely equilateral triangles are erected on the six sides of a plane oriented hexagon, then one set of three alternate vertices forms a directly equilateral triangle if and only if the other set does so.*

The degenerate case of this corollary which appears when the "hexagon" is a triangle with an additional vertex on each side can be expressed as follows.

THEOREM 5 (Garfunkel and Stahl). *If L, M, N are points on the sides BC, CA, AB of a triangle ABC, and if the vertices of inversely equilateral triangles erected on the alternate segments LC, MA, NB form a directly equilateral triangle, then the vertices of inversely equilateral triangles erected on the other alternate segments BL, CM, AN also form a directly equilateral triangle.*

This is a more precise statement of Theorem 7 of [**1**], which ignores the vital question of the orientation of the triangles. Clearly Theorem 4 is a generalization of Theorem 5; a completely different generalization of Theorem 5 is given in Theorem 8 below.

Theorem 4, and all the theorems to follow, admit numerous corollaries obtained by allowing one or more of the triangles in a figure to degenerate to a point. In the corollary to Theorem 4, for example, we may allow the last two triangles to degenerate. On reflection of the figure we then obtain the following result.

Of the six oriented triangles obtained by joining the sides of an oriented plane hexagon alternately to two coplanar points, if five are directly equilateral, then so is the sixth.

6. Primitive similarity systems.

THEOREM 6. *If θ, ϕ are any two permutations on the set $\{1, 2, \cdots, k\}$, then the configuration $\{A_rB_rC_r, A_rB_{\theta r}C_{\phi r}: r=1, 2, \cdots, k\}$, of $2k$ triangles sharing $3k$ vertices, is a similarity system.*

Proof. If all but one of the $2k$ triangles are directly similar then, for some x, y, z not all equal, the coordinates of their vertices satisfy all but one of the $2k$ conditions

$$(5)\qquad \begin{aligned} (y-z)a_i + (z-x)b_i + (x-y)c_i &= 0, \\ (y-z)a_i + (z-x)b_{\theta i} + (x-y)c_{\phi i} &= 0, \end{aligned} \qquad (i = 1, 2, \cdots, k).$$

Now, whatever the values of x, y, z, the sum of the left hand sides of the first k of these conditions is the same as the sum of the left hand sides of the remaining k conditions. Hence, whatever the shape of the triangles, the direct similarity of $2k-1$ of them implies the direct similarity of them all, and this completes the proof of Theorem 6.

DEFINITION. *The similarity system described in Theorem 6 will be called a primitive similarity system.*

If any of the permutations θ, ϕ, $\phi^{-1}\theta$ has a fixed point, Theorem 6 takes a degenerate form. The nondegenerate case $k=3$ is illustrated in Figure 2, which shows six triangles

$$(6)\qquad \{A_1B_1C_1,\ A_2B_2C_2,\ A_3B_3C_3,\ A_1B_2C_3,\ A_2B_3C_1,\ A_3B_1C_2\}$$

whose incidence relations are such that if (as in the figure) any five are directly similar, then so is the sixth.

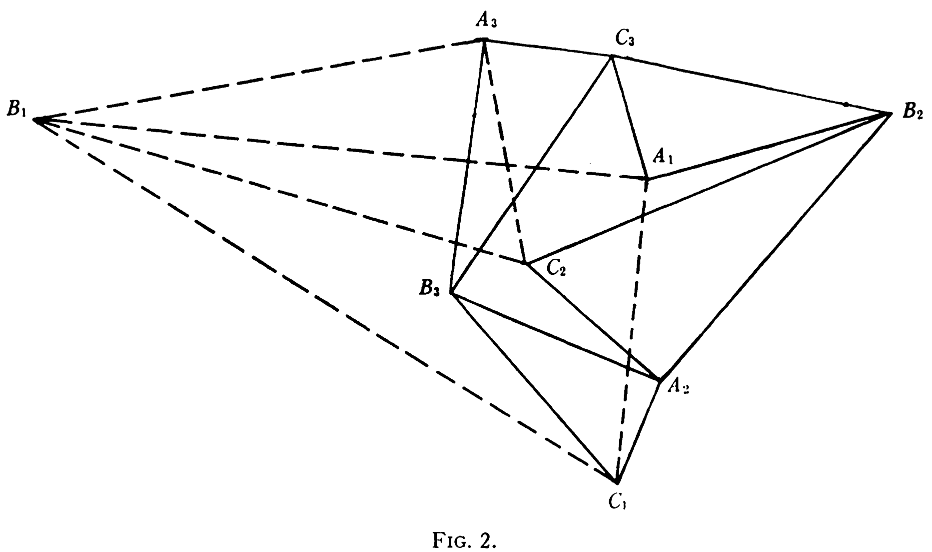

FIG. 2.

If in Theorem 6 we take $k=4$, $\theta(1, 2, 3, 4)=(4, 3, 2, 1)$ and $\phi(1, 2, 3, 4)=(3, 4, 1, 2)$ we obtain Theorem 4. Taking $k=4$ and θ, ϕ cyclic permutations, we obtain a primitive similarity system essentially different from that described in Theorem 4, namely the configuration

(7) $\{A_1B_1C_1,\ A_2B_2C_2,\ A_3B_3C_3,\ A_4B_4C_4,\ A_1B_2C_3,\ A_2B_3C_4,\ A_3B_4C_1,\ A_4B_1C_2\}.$

This configuration, illustrated in Figure 3, is most easily distinguished from the configuration (4) by the fact that it contains quadrilaterals, such as $A_1B_1C_2B_2$, with vertices from each of the three positions in a (hyper-oriented) triangle.

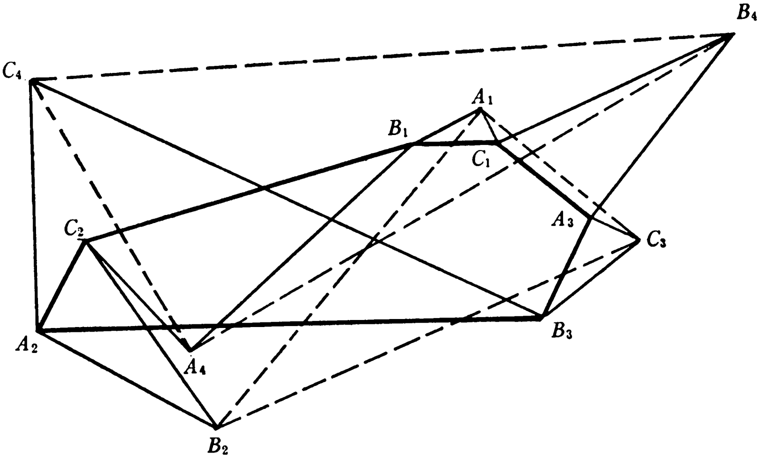

FIG. 3.

Regarding $A_3B_3A_2C_2B_1C_1$ as an oriented hexagon and rewriting A_1, B_4, C_3, C_4, B_2, A_4 as A, B, C, D, E, F, we obtain the following analogue of the corollary to Theorem 4.

THEOREM 7. *If A, B, C are the vertices (in order) of directly equilateral triangles erected on three consecutive sides of a plane oriented hexagon, and if D, E, F (in order) are the vertices of inversely equilateral triangles erected on the other three sides, then the triangle BDF is directly equilateral if and only if the triangle ACE is inversely equilateral.*

From primitive similarity systems we may obtain further similarity systems by adding to the configuration further incidence relations (that is, by identifying points in the figure), and it may well be the case that all similarity systems can be obtained in this manner. For example, the two similarity systems mentioned in Section 4 are obtained from the primitive similarity system (4) by taking, respectively,

(8) $A_1=A,\ B_1=B,\ C_1=C,\ A_2=B_2=C_2=P,\ A_3=B_3=C_3=Q,\ A_4=B_4=C_4=R;$

(9) $A_1=A,\ B_1=B,\ C_2=C,\ A_2=D,\ B_3=E,\ C_3=F,\ A_3=A_4=B_2=B_4=C_1=C_4=O.$

It would be interesting to know whether there is a three-dimensional analogue of a similarity system. The topic may be regarded as a branch of metric graph theory.

7. Triangles associated with hexagons.

THEOREM 8. *Let K be any point other than Y or Z in the plane of a directly equilateral triangle XYZ whose centre is O. Let $A_1A_2A_3A_4A_5A_6(A_7=A_1)$ be a plane hexagon and let P_1, P_2, P_3, Q_1, Q_2, Q_3 be coplanar points such that, for each i, the triangles $P_iA_{2i}A_{2i-1}$ and $Q_iA_{2i}A_{2i+1}$ respectively are directly similar to the triangles OYK, OZK. Then the triangle $P_1P_2P_3$ is directly equilateral if and only if the triangle $Q_1Q_2Q_3$ is directly equilateral.*

Proof. Taking $x=1$, $y=\omega$, $z=\omega^2$, so that O is represented by zero, we have, from (1),

$$\text{(10)} \qquad \begin{aligned} (\omega - k)p_i + ka_{2i} - \omega a_{2i-1} &= 0, \\ (\omega^2 - k)q_i + ka_{2i} - \omega^2 a_{2i+1} &= 0, \end{aligned} \qquad (i = 1, 2, 3).$$

Then the condition that $P_1P_2P_3$ is directly equilateral can be written $(\omega-k)(p_1+\omega p_2+\omega^2 p_3)=0$ and the condition that $Q_1Q_2Q_3$ is directly equilateral can be written $(\omega^2-k)(q_1+\omega q_2+\omega^2 q_3)=0$. By virtue of (10), either of these conditions can be written

$$\text{(11)} \qquad \omega a_1 - ka_2 + \omega^2 a_3 - k\omega a_4 + a_5 - k\omega^2 a_6 = 0$$

and this completes the proof of Theorem 8, which is generalized later in Theorem 9.

Various interesting special cases of Theorem 8 arise from particular choices of the point K. For example, letting K coincide with X, we obtain the

COROLLARY. *If directly equilateral triangles are erected on the middle thirds of the six sides of a plane oriented hexagon, then one set of three alternate vertices forms a directly equilateral triangle if and only if the other set does so.*

This corollary may be compared with "Napoleon's Theorem."

If we take K such that O is the midpoint of YK we obtain Theorem 1, and if we take K such that O is the midpoint of XK we obtain the corollary to Theorem 4. This last, as we have seen, implies Theorem 5 and it follows that Theorem 8, illustrated in Figure 5, is a generalization of Theorem 5. Figure 5 shows two triads of similar triangles and two equilateral triangles, while Figure 4 (illustrating Theorem 4) shows a set of eight similar triangles. Theorem 5 is the particular case of either of these figures in which all eight triangles are equilateral (and in which also the central hexagon degenerates to a triangle).

8. Medially conjugate triangles. If ABC is a triangle in the complex plane whose vertices are represented by the complex numbers a, b, c, then the complex numbers $b+c-2a$, $c+a-2b$, $a+b-2c$ represent vectors parallel to the medians of the triangle ABC and, since their sum is zero, they also represent the (vector) sides of a triangle. Thus the condition that the triangle XYZ should be directly similar to a triangle with sides parallel to the medians of the triangle ABC is

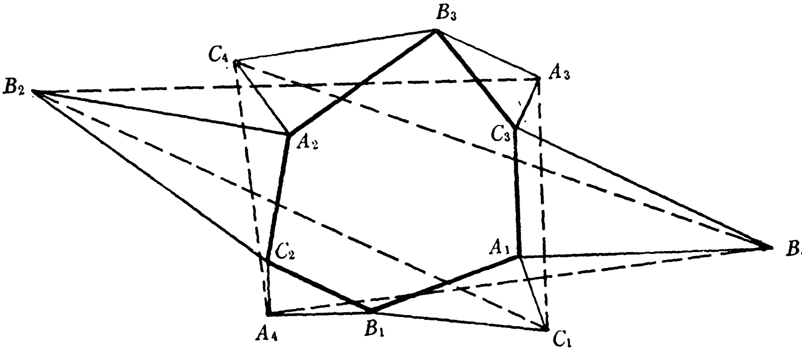

FIG. 4.

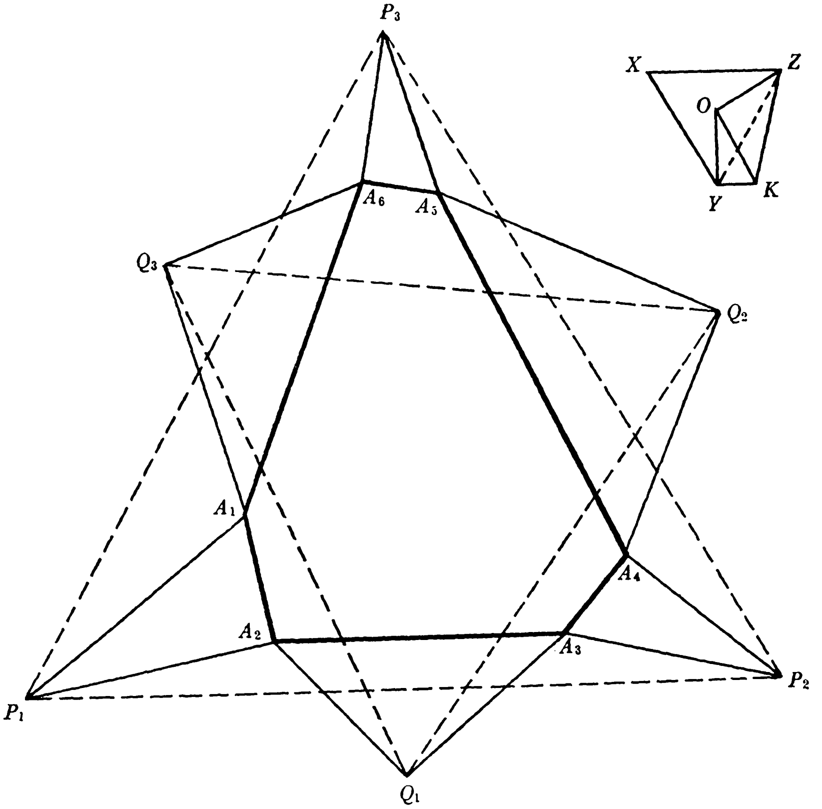

FIG. 5.

$(z-x)/(x-y)=(c+a-2b)/(a+b-2c)$, which can be written in the symmetric form

$$3(ax+by+cz)=(a+b+c)(x+y+z). \tag{12}$$

This leads us to the

DEFINITION. *Two triangles will be said to be medially conjugate if one of them (and therefore each of them) is directly similar to a triangle with sides parallel to the medians of the other. In particular the medial conjugate of a directly equilateral triangle is a directly equilateral triangle.* We are now in a position to state

THEOREM 9. *Let* $A_1A_2A_3A_4A_5A_6$ *and* $B_1B_2B_3B_4B_5B_6$ *be two coplanar hexagons and let* $\{B_i'\}$ *be the hexagon obtained by rotating* $\{B_i\}$ *through two right angles about the centroid G of its vertices* B_i. *Let* $P_1, P_2, P_3, Q_1, Q_2, Q_3$ *be coplanar points such that, for each i, the triangles* $P_iA_{2i}A_{2i-1}$ *and* $Q_iA_{2i}A_{2i+1}$ *respectively are directly similar to the triangles* $GB'_{2i-1}B_{2i}$, $GB'_{2i+1}B_{2i}(A_7=A_1,\ B_7'=B_1')$, *and let* P_i', Q_i' $(i=1, 2, 3)$ *be the midpoints of* $B_{2i-1}B_{2i}$, $B_{2i+1}B_{2i}(B_7=B_1)$. *Then the triangles* $P_1P_2P_3$, $P_1'P_2'P_3'$ *are medially conjugate if and only if the triangles* $Q_1Q_2Q_3$, $Q_1'Q_2'Q_3'$ *are medially conjugate.*

Theorem 8 is the particular case of Theorem 9 in which $B_1B_3B_5$ and $B_2B_4B_6$ are concentric directly equilateral triangles.

Proof of Theorem 9. Taking, as we may, $\sum b_i=0$, we have $\sum p_i' = \sum q_i' = 0$, $g=0$, and $b_i' = -b_i$. Substituting $2p_i'$ for $b_{2i-1}+b_{2i}$ and $2q_i'$ for $b_{2i+1}+b_{2i}$, we find from (1),

$$
\begin{aligned}
2p_i'p_i &= a_{2i-1}b_{2i-1} + a_{2i}b_{2i}, \\
2q_i'q_i &= a_{2i+1}b_{2i+1} + a_{2i}b_{2i},
\end{aligned}
\qquad (i = 1, 2, 3). \tag{13}
$$

Hence, from (12), the condition for medial conjugacy of the triangles $P_1P_2P_3$, $P_1'P_2'P_3'$ is the same as that for the medial conjugacy of the triangles $Q_1Q_2Q_3$, $Q_1'Q_2'Q_3'$, namely

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 = 0 \qquad \left(\sum b_i = 0\right). \tag{14}$$

In the general case, when $\sum b_i \neq 0$, this relation becomes

$$6 \sum a_ib_i = \sum a_i \sum b_i, \tag{15}$$

which may be compared with (12). The relation (15) between the hexagons $\{A_i\}$, $\{B_i\}$ is symmetric and is also invariant under simultaneous identical permutations of the vertices. Noticing that, for example, the permutation which interchanges the first two vertices does not change $P_1P_2P_3$ or $P_1'P_2'P_3'$, we find that, associated with a general pair of hexagons $\{A_i\}$, $\{B_i\}$, there are thirty distinct pairs of unoriented triangles (i.e., 180 distinct pairs of triangles) such as $P_1P_2P_3$, $P_1'P_2'P_3'$, of which every pair is medially conjugate if one pair is so.

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LOGICAL PARADOXES ARE ACCEPTABLE IN BOOLEAN ALGEBRA

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1. Introduction. Axioms for the calculus of propositions in logic can be given in several ways, one [1] of which is the following:

1. $[p\supset q]\supset[q\supset r\supset[p\supset r]]$
2. $[\sim p\supset p]\supset p$
3. $p\supset[\sim p\supset q]$.

In words, these axioms can be stated, loosely, as follows:

1. *If p implies q ($p\supset q$), and q implies r, then p implies r; this axiom is well known from Aristotelian logic: "All men are mortal, Socrates is a man, therefore Socrates is mortal."*

2. *If by assuming "p is false" ($\sim p$) we can deduce that "p is true" then "p is false" cannot be true, hence p is true. It represents the "reductio ad absurdum."*

3. *p being true implies that "not p" implies any statement to be true: If it is true that Columbus discovered America, then Columbus not having discovered America implies that Paris is in Japan!*

Number 3 may be necessary as an axiom, but in words it strikes one as rather queer. (For a discussion of this "paradox of material implication" see for instance [**6**, p. 54].) 2 is also criticized, by the intuitionist mathematicians, who replace it [1] by the weaker one:

2′. $[p\supset\sim p]\supset\sim p$.

When we compare 2′ with 2, we see that the only difference is that the p and $\sim p$ have changed place. This is significant only because the intuitionists do not admit that a proposition *must* be either true or false (*"tertium no datur"*). They say that to assume this is nothing more than a habit of thought, which usually is justified. However, to consider this as a necessary law of thought under all circumstances is a dangerous thing. In particular, when dealing with an infinite set of objects, like the whole numbers, they will only admit a proposition to be either true or false if a method can be constructed to check the proposition for all members in a finite operation.

There is possible a notation [2] for the Boolean algebra which avoids our intuitive dislike for these axioms, and yet is not inferior to the notation used above. In fact, it is more convenient, since it is practically identical with conventional algebra. In addition, it also allows the circumvention of the law of the excluded middle 2, by following a familiar path of conventional algebra.

2. The algebra modulo two. The notation in question is based on arithmetic modulo two:

$$\text{odd} + \text{odd} = \text{even};\ \text{odd} + \text{even} = \text{odd};\ \text{even} + \text{even} = \text{even};$$
$$\text{odd}\cdot\text{odd} = \text{odd};\ \text{odd}\cdot\text{even} = \text{even};\ \text{even}\cdot\text{even} = \text{even}.$$

Indicating "odd" by "1" and "even" by "0", we have:

$$1+1=0;\ 1+0=1;\ 0+0=0$$

$$1\cdot 1=1;\ 1\cdot 0=0;\ 0\cdot 0=0.$$

To make this a calculus of propositions, we identify "1" with "true" and "0" with "false." Every proposition is now a formula in modulo two algebra. $\sim p$, not p, is written as $p+1$, and "it is true that $p\supset q$," becomes: $p+pq+1=1$. Substituting $p=1$ gives $q=1$ (since $p+p=0$) and for $p=0$, $0\cdot q=0$, so q can either be 1 or 0.

(The present notation is very close to the one used by Boole himself [**3**]. However, Boole used the "+" sign for "p or q or both," the inclusive disjunction, often written as $p\cup q$, which in our notation would be $p+q+pq$. Also, $\sim p$ is written by him as $1-p$ instead of $1+p$. No doubt, this notation was the more obvious choice for Boole, because at that time the notation for finite fields had not yet been developed.)

The original axioms 1, 2, and 3 have now become theorems, since as axioms they have been replaced by the well-known axioms of the algebra of a finite field. For instance, axiom 1 becomes the theorem:

1. $A+A[B+BC+1]+1=1$ *in which* $A=p+pq+1$; $B=q+qr+1$, $C=p+pr+1$.

The theorem is readily proved to be a tautology by the repeated application of $x+x=0$ and $x^2=x$.

In the same way, we have Theorems 2 and 3:

2. $D+Dp+1=1$; $D=(1+p)+(1+p)p+1$.

3. $p+pE+1=1$; $E=(1+p)+(1+p)q+1$.

The proofs are again straightforward.

It is particularly noteworthy that the original axiom 2 is proved as a theorem. However, there is a natural way of getting around the law of the excluded middle in our formalism. This is done by the general method of extending [**4**] a field by the roots of an equation, $x^2=x$, which in fact does say "*tertium non datur*," is not a standard axiom of the field modulo two, as is $x+x=0$. It is more like Euclid's parallel postulate in geometry. Its modification leads to a new logical scheme which still retains the old one in a central position.

3. Extension of the field. We start with the classic paradox of Eubulides: "This statement (which I now make) is false." This is a paradox, because if the statement is true, it is false, therefore it is true, etc. Call the statement between quotation marks "x." In the quotation marks it is stated that x is false, or $x+1$ is true. Can both x and $x+1$ be true? We remember that $p\cdot q$ stands for "both p and q," and write:

$$x(x+1)=1 \quad \text{or } x^2+x+1=0.$$

Now this equation is not satisfied either by $x=1$ or by $x=0$, and that of course is what makes it a paradox. However, we can follow the conventional algebraic method [**4**] of *defining* new numbers which are the roots of this equation and then extending the field $\{0, 1\}$ with these roots, which we call α_1 and α_2. This gives us the identity

$$x^2 + x + 1 = (x + \alpha_1)(x + \alpha_2) = x^2 + (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2,$$

or

$$\alpha_1 + \alpha_2 = 1; \qquad \alpha_1\alpha_2 = 1.$$

There exists a simple realization of these roots by writing two by two commutable matrices instead of numbers:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad \alpha_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}; \quad \alpha_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is readily checked that α_1 and α_2 satisfy the equation $x^2+x+1=0$, by substitution and applying the laws of matrix multiplication and addition:

$$\alpha_1^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\alpha_1^2 + \alpha_1 + 1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

The extension of the field $\{0, 1\}$ contains only the four numbers $(0, 1, \alpha_1, \alpha_2)$; no other numbers are generated by addition, multiplication, or division. It [4] is the Galois field GF (2^2). For instance

$$\alpha_1^2 = \alpha_2; \quad \alpha_2^2 = \alpha_1; \quad \alpha_1\alpha_2 = \alpha_2\alpha_1 = \alpha_1^3 = \alpha_2^3 = 1; \quad \alpha_1 + 1 = \alpha_2.$$

It should be remarked that with this extension, it becomes attractive to write "p implies q" as: p^2+pq+1. This makes "p implies p" reducible to the axiom $x+x=0$. In making this change, the original axiom 2 is still a tautology. However, 1 and 3 can no longer be made into tautologies; they are only true if $x^2=x$. The resulting algebra may be considered to belong in a general category of multi-valued logics, the first of which was proposed by Post [6]. However, there does not exist a one to one correspondence between the "Post algebra" and the present scheme.

4. Discussion. In number theory we have become quite used to logical extensions of the set of numbers, whenever the operations on numbers make this desirable. When we are asked to divide five loaves of bread equally among two people, the answer is $2\frac{1}{2}$, because fractions of loaves are acceptable. When we are asked to divide five people equally over two rooms, the problem has no solution, because $2\frac{1}{2}$ is not a whole number. It would be a paradox only if we assumed that every problem in whole numbers *had* to have a solution. Similarly, in ancient Greek mathematics, the problem of determining a/b such that $(a/b)^2=2$, formed a paradox [5], because it could be proved that a and b could be neither even nor odd. Modern mathematics has solved this paradox by introducing irrational numbers.

In logic as applied to mathematics [6], it has been Russell's viewpoint in the Principia Mathematica that a paradox of the classical type has to be avoided at all cost: when a statement *must* be either true or false ($p^2+p=0$), then any paradox leads to $1=0$, true is false, and every theorem as well as its negation is true. However, Goedel's proof [1] has shown that even in the theory of whole numbers there is a well-formed formula A such that neither "A is true" nor "A is false" is a theorem. It seems clear that a paradox of the classical type is hard to avoid, both in ordinary language and in rigorous mathematics. From the algebraic point of view, it seems natural to allow these paradoxes to be present and to say that they correspond to an extension of the field.

It is interesting that Boole already realized [3] the theoretical possibility of extending his logic by considering $x^3=x$ instead of $x^2=x$, and that this would lead to different laws: "But they are of a nature altogether foreign to the province of general reasoning." It was Brouwer [7] who first took issue with this classical point of view in the case of infinite sets, thus formulating intuitionism.

Is there any practical significance to the roots α_1 and α_2? The answer, of course, falls outside the realm of pure mathematics. Imaginary numbers were defined three centuries [5] before they became the indispensable tools of the electrical engineer, in network theory, and of the physicist, in quantum mechanics. Logic, after all, has nothing to say about reality, but only about the human way of approaching it.

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LINEARIZATION TRANSFORMATIONS FOR LEAST SQUARES PROBLEMS

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In various elementary courses of mathematics and statistics, the student is introduced to least squares curve fitting. After learning to obtain linear least squares fits, he is almost invariably presented with the form $y=ae^{bx}$ and told that by application of the transformation $z=\ln y$ he can reduce the least squares problem associated with this exponential form to a linear least squares problem. What he is seldom told, however, is that the application of the logarithmic transformation distorts his scale so that minimization of $\sum(\ln y_i-(\ln a+bx_i))^2$ is not equivalent to minimization of $\sum(y_i-ae^{bx_i})^2$. This observation is not new, see [1], p. 195, for example, but it has been neglected to the extent that the

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Form	*Linearization Transformation*
$y=ae^{bx}$	$z=\ln y$
$y=ax^b$	$z=\ln y$
$y=\ln(a+bx)$	$z=e^y$
$y=\{a^2+b^2x^2\}^{1/2}$	$z=y^2$
$y=a(x-b)^2$	$z=\sqrt{y}$
$y=a/(b-x)$	$z=1/y$
$y=ax/(b-x)$	$z=1/y$

For forms such as these, the use of linearization transformations is both computationally more efficient and aesthetically more satisfying than the use of iterative techniques, such as the method of differential corrections and the Newton-Raphson method, to solve the nonlinear normal equations. Of course, the accuracy obtainable with linearization transformations is not as good as that obtainable with the iterative techniques; but, even when very high accuracy is required, linearization transformations are of value in providing good initial estimates for the iterative techniques. The purpose of this note, then, is to establish, for the undergraduate, a theoretical framework within which the proper application of linearization transformations can be justified.

For conciseness, we will consider forms $y=f(x, a, b)$ which involve only two parameters to be determined by least squares. The results are perfectly general, however, and their extension to any number of parameters is obvious. Suppose, then, that we are given the form $y=f(x, a, b)$ and a set of data points $\{(x_i, y_i)\}_{i=1}^n$, $n>2$. We consider the least squares problem of minimizing the function

$$S(a, b) = \sum_{i=1}^{n} \{y_i - f(x_i, a, b)\}^2.$$

It is assumed that there is a set $X \subset R_1$, with $x_i \in X$ $(i=1, \cdots, n)$, and an open set $D \subset R_2$ such that f is a function from $X \times D$ to R_1 and S is a function from D to R_1 (where R_k denotes the set of all k-tuples of real numbers, with the Euclidean metric topology). For each i $(i=1, \cdots, n)$, the partial derivatives of $f(x_i, a, b)$ with respect to a and b are assumed to exist at all points of D. Let Y

be a connected subset of R_1 which contains the range of f and the numbers $y_1, \cdots, y_n$ from the data. A function $g: Y \to R_1$ is said to be a *linearization transformation* for the form $y=f(x, a, b)$ provided there exist functions P, Q, R, from X to R_1, and functions A, B, from D to R_1, such that for all $x \in X$ and all $(a, b) \in D$

$$g[f(x, a, b)] = A(a, b)P(x) + B(a, b)Q(x) + R(x)$$

and such that the Jacobian $\partial(A, B)/\partial(a, b)$ is nonvanishing in D. For example, consider the form $y=a(x-b)^2$, and let $m=$ smallest x_i in the set of data. Let $X=\{x: x \geqq m\}$ and let $D=\{a: a>0\} \times \{b: b<m\}$. Let $f(x, a, b)=a(x-b)^2$ for all $x \in X$ and all $(a, b) \in D$. Suppose all the y_i's are positive, and let $Y=\{y: y>0\}$. Then $g(y)=y^{1/2}$ is a linearization transformation for the form $y=f(x, a, b) =a(x-b)^2$, since $g[a(x-b)^2]=Ax+B$ where $A=A(a, b)=\sqrt{a}$ and $B=B(a, b) =-b\sqrt{a}$ for all (a, b) in D, and $\partial(A, B)/\partial(a, b)=-1/2$. Here, of course, P leaves all points of X fixed, Q maps all points of X to 1, and R maps all points of X to 0.

We return now to the general case. If g is a linearization transformation for the form $y=f(x, a, b)$ then, for any given set of numbers $w_1, \cdots, w_n$, one can consider the least squares problem of determining a and b so that the function

$$\begin{aligned} T(a, b; w_1, \cdots, w_n) &= \sum_{i=1}^{n} w_i\{g[y_i] - g[f(x_i, a, b)]\}^2 \\ &= \sum_{i=1}^{n} w_i\{g[y_i] - [A(a, b)P(x_i) + B(a, b)Q(x_i) + R(x_i)]\}^2 \\ &= H(A, B; w_1, \cdots, w_n) \end{aligned}$$

will be minimized. For each set of numbers $w_1, \cdots, w_n$, this associated least squares problem is a *weighted linear least squares problem* in terms of the parameters A and B, so that the normal equations $\partial H/\partial A=\partial H/\partial B=0$ are linear and can be solved for A and B by the usual methods for linear systems. One can then obtain a and b by simultaneous solution of the equations $A(a, b)=A$, $B(a, b)=B$, since $\partial(A, B)/\partial(a, b) \neq 0$.

THEOREM. *Suppose $S(a, b)$ is minimized at the point (a_0, b_0) in D. If the linearization transformation g has a nonzero derivative at each point of Y, then there exist numbers $w_1, \cdots, w_n$ such that $T(a, b; w_1, \cdots, w_n)$ is minimized at (a_0, b_0).*

Proof. We have $\partial S/\partial a=\partial S/\partial b=0$ at (a_0, b_0), whence

$$\sum_{i=1}^{n} (y_i - f(x_i, a_0, b_0))f_a(x_i, a_0, b_0) = 0$$

$$\sum_{i=1}^{n} (y_i - f(x_i, a_0, b_0))f_b(x_i, a_0, b_0) = 0.$$

For any numbers $w_1, \cdots, w_n$ we have

$$T_a(a_0, b_0; w_1, \cdots, w_n)$$
$$= -2\sum_{i=1}^{n} w_i\{g[y_i] - g[f(x_i, a_0, b_0)]\}g'[f(x_i, a_0, b_0)]\cdot f_a(x_i, a_0, b_0).$$

By the mean value theorem, for each i ($i=1, \cdots, n$) there exists a point ξ_i between y_i and $f(x_i, a_0, b_0)$ such that

$$g[y_i] - g[f(x_i, a_0, b_0)] = g'(\xi_i)\{y_i - f(x_i, a_0, b_0)\}$$

and so we see that we will have $T_a(a_0, b_0; w_1, \cdots, w_n)=0$ provided we set $w_i=1/\{g'(\xi_i)g'[f(x_i, a_0, b_0)]\}$, $i=1, \cdots, n$. It is clear that this same set of w_i's will make $T_b(a_0, b_0; w_1, \cdots, w_n)=0$. Now since

$$\frac{\partial T}{\partial a} = \frac{\partial H}{\partial A}\cdot\frac{\partial A}{\partial a} + \frac{\partial H}{\partial B}\cdot\frac{\partial B}{\partial a}, \qquad \frac{\partial T}{\partial b} = \frac{\partial H}{\partial A}\cdot\frac{\partial A}{\partial b} + \frac{\partial H}{\partial B}\cdot\frac{\partial B}{\partial b},$$

and since the Jacobian $\partial(A, B)/\partial(a, b)\neq 0$ at (a_0, b_0), we see that $\partial H/\partial A = \partial H/\partial B=0$ at $A_0=A(a_0, b_0)$, $B_0=B(a_0, b_0)$. But it is well known that the linear normal equations $\partial H/\partial A = \partial H/\partial B=0$ have a unique solution and that this solution does indeed correspond to the minimum of the function $H(A, B; w_1, \cdots, w_n)=T(a, b; w_1, \cdots, w_n)$.

From a practical standpoint, one must, of course, use estimates of these weights $w_i=1/\{g'(\xi_i)g'[f(x_i, a_0, b_0)]\}$ in the linearization procedure. Since it is expected that a_0 and b_0 will turn out such that $f(x_i, a_0, b_0)$ will be fairly close to y_i, for $i=1, \cdots, n$, and since ξ_i must be between $f(x_i, a_0, b_0)$ and y_i, it is reasonable to use $w_i^*=1/\{g'(y_i)\}^2$ as an estimate of w_i provided g' is continuous. One can then solve the weighted linear least squares problem, using the weights w_i^*, to obtain A_0^*, B_0^*; and a_0^*, b_0^* are then obtained by simultaneous solution of the equations $A(a_0^*, b_0^*)=A_0^*$, $B(a_0^*, b_0^*)=B_0^*$. The remaining problem is to estimate upper bounds for $|\Delta a_0^*|=|a_0^*-a_0|$ and $|\Delta b_0^*|=|b_0^*-b_0|$. We have

$$\Delta w_i^* = w_i^* - w_i = \frac{1}{\{g'(y_i)\}^2} - \frac{1}{g'(\xi_i)g'(f(x_i, a_0, b_0))}.$$

Assuming g' to be monotonic, (which will generally be the case in applications) we have

$$|\Delta w_i^*| \leqq \left|\frac{1}{\{g'(y_i)\}^2} - \frac{1}{\{g'(f(x_i, a_0, b_0))\}^2}\right|.$$

If we assume the existence of g'' at all points of Y (again a plausible assumption), then the mean value theorem gives us the existence of points η_i between y_i and $f(x_i, a_0, b_0)$ such that

$$|\Delta w_i^*| \leqq \left|\frac{-2g''(\eta_i)}{\{g'(\eta_i)\}^3}\{y_i - f(x_i, a_0, b_0)\}\right|.$$

Finally, assuming $|2g''/(g')^3| \leqq M$ on Y (or, at least, on a connected subset of Y containing the points y_i and $f(x_i, a_0, b_0)$ $i=1, \cdots, n$), we have

$$|\Delta w_i^*| \leqq M \cdot |y_i - f(x_i, a_0, b_0)|$$

and so

$$\sum_{i=1}^{n} (\Delta w_i^*)^2 \leqq M^2 \sum_{i=1}^{n} \{y_i - f(x_i, a_0, b_0)\}^2 \leqq M^2 \sum_{i=1}^{n} \{y_i - f(x_i, a_0^*, b_0^*)\}^2$$

since $S(a, b)$ is minimized at (a_0, b_0).

Differentiating the normal equations $\partial H/\partial A = \partial H/\partial B = 0$ partially with respect to w_i^* yields equations which are linear in $\partial A_0^*/\partial w_i^*$ and $\partial B_0^*/\partial w_i^*$. Hence, we get

$$\frac{\partial A_0^*}{\partial w_i^*} = k_i\{g[y_i] - g[f(x_i, a_0^*, b_0^*)]\} \quad \text{and} \quad \frac{\partial B_0^*}{\partial w_i^*} = l_i\{g[y_i] - g[f(x_i, a_0^*, b_0^*)]\},$$

where

$$k_i = \frac{P(x_i)\sum_{j=1}^{n} w_j^* Q(x_j)^2 - Q(x_i)\sum_{j=1}^{n} w_j^* P(x_j)Q(x_j)}{\left(\sum_{j=1}^{n} w_j^* P(x_j)^2\right)\left(\sum_{j=1}^{n} w_j^* Q(x_j)^2\right) - \left(\sum_{j=1}^{n} w_j^* P(x_j)Q(x_j)\right)^2}$$

and

$$l_i = \frac{Q(x_i)\sum_{j=1}^{n} w_j^* P(x_j)^2 - P(x_i)\sum_{j=1}^{n} w_j^* P(x_j)Q(x_j)}{\left(\sum_{j=1}^{n} w_j^* P(x_j)^2\right)\left(\sum_{j=1}^{n} w_j^* Q(x_j)^2\right) - \left(\sum_{j=1}^{n} w_j^* P(x_j)Q(x_j)\right)^2}.$$

Finally, we have

$$\Delta A_0^* \approx \sum_{i=1}^{n} \frac{\partial A_0^*}{\partial w_i^*} \Delta w_i^*$$

so that, using the Cauchy-Schwarz inequality

$$(\Delta A_0^*)^2 \leqq \left\{\sum_{i=1}^{n} \left(\frac{\partial A_0^*}{\partial w_i^*}\right)^2\right\} \cdot \left\{\sum_{i=1}^{n} (\Delta w_i^*)^2\right\},$$

whence

$$(\Delta A_0^*)^2 \leqq \left(\sum_{i=1}^{n} k_i^2\{g[y_i] - g[f(x_i, a_0^*, b_0^*)]\}^2\right)\left(M^2 \sum_{i=1}^{n} \{y_i - f(x_i, a_0^*, b_0^*)\}^2\right).$$

Similarly, we obtain

$$(\Delta B_0^*)^2 \leqq \left(\sum_{i=1}^{n} l_i^2 \{ g[y_i] - g[f(x_i, a_0^*, b_0^*)] \}^2 \right) \left(M^2 \sum_{i=1}^{n} \{ y_i - f(x_i, a_0^*, b_0^*) \}^2 \right).$$

Bounds for Δa_0^* and Δb_0^* can now be obtained by examining the transformations $A(a, b)$, $B(a, b)$ to see what region of the ab-plane corresponds to the region $A_0^* \pm |\Delta A_0^*|_{\max}$, $B_0^* \pm |\Delta B_0^*|_{\max}$ of the AB-plane.

As a numerical example, consider fitting the three data points $(x_i, y_i) = (3, 0.2), (4, 2.3), (5, 4.4)$, with the form $f(x, a, b) = a(x-b)^2$ discussed above. One may easily check that an exact solution of the normal equations is $a_0 = .500$, $b_0 = 2.00$, and that this solution does indeed minimize $S(a, b)$. Linearization is accomplished by the transformation $g[y] = y^{1/2}$, and we have $A = \sqrt{a}$, $B = -b\sqrt{a}$, $P(x) = x$, $Q(x) = 1$, $R(x) = 0$. Using the weights $w_i^* = 1/\{g'[y_i]\}^2 = 4y_i$ we obtain $A_0^* = .6579$ and $B_0^* = -1.176$, whence $a_0^* = (A_0^*)^2 = .433$ and $b_0^* = -B_0^*/A_0^* = 1.79$. Noting that $-2g''/(g')^3 \equiv 4$, we compute error bounds as indicated above: $|\Delta A_0^*| \leqq .1317$, $|\Delta B_0^*| \leqq 0.631$. Hence A_0 should be in the interval $[.5262, .7896]$, and B_0 should be in the interval $[-1.807, -0.545]$. The actual values of A_0, B_0 are, of course, $A_0 = .7071$ and $B_0 = -1.414$. The inverse transformation equations $a = A^2$, $b = -B/A$ now give us at once the following bounds for a_0 and b_0: $.277 \leqq a_0 \leqq .623$, $0.69 \leqq b_0 \leqq 3.43$. It is interesting to observe that the usual nonrigorous linearization (accomplished with the same transformation but with all weights set equal to 1) yields $A_0^{**} = .8252$, $B_0^{**} = -1.947$, whence $a_0^{**} = (A_0^{**})^2 = .681$ and $b_0^{**} = -B_0^{**}/A_0^{**} = 2.36$. We note that $|a_0^{**} - a_0| > |a_0^* - a_0|$ and $|b_0^{**} - b_0| > |b_0^* - b_0|$, and that the error bounds obtained above for a_0 actually exclude the value $a_0^{**} = .681$. Finally, we have $S(a_0, b_0) = .190$, $S(a_0^*, b_0^*) = .226$, and $S(a_0^{**}, b_0^{**}) = .345$.

It is to be noted that our error bounds for A_0^* and B_0^* are, in fact, bounds for the total differentials of A_0^* and B_0^* with respect to the w_i^*, $i = 1, \cdots, n$. These bounds are therefore approximate. We recall, however, that $|\Delta w_i^*| \leqq M \cdot |y_i - f(x_i, a_0, b_0)|$ so that if a good fit of the data is possible with the form $y = f(x, a, b)$ and if $|2g''/(g')^3|$ is not too large, then the Δw_i^* will be small and our error bounds should be valid. In practice, an indication of the validity of the bounds can be obtained by observing the size of $M^2 \sum_{i=1}^{n} \{y_i - f(x_i, a_0^*, b_0^*)\}^2$ which is an upper bound for $\sum_{i=1}^{n} (\Delta w_i^*)^2$.

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BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Elementary Introduction to Number Theory. By Calvin T. Long. Heath, Boston, Massachusetts, 1965. x+150 pp. $4.20.

According to the author's preface this book is designed for use in a one-semester course in number theory at a level somewhat lower than that of the usual college course. The material has been used in courses taught to mathematics and mathematics-education majors at the sophomore-junior level, and to participants in NSF Summer Institutes. In dealing with this material vis-à-vis the intended audience, the author has made a serious attempt to give his proofs in considerable detail and to organize his material to display the essential structure of the subject, stressing methods rather than tricks in his proofs.

In Chapter 1, besides presenting the notational and postulational basis of the subject, the author discusses mathematical induction, well ordering, positional notation, and the division algorithm. Chapter 2 deals with divisibility properties, greatest common divisor, least common multiple, the Euclidean algorithm, the fundamental theorem of arithmetic, and Pythagorean triplets. Chapter 3 is devoted to prime numbers, and covers the sieve of Eratosthenes as well as Mersenne and Fermat numbers. It also contains a discussion of the prime number theorem, without proofs. Chapters 4 and 5 deal with congruences and include such topics as reduced residue systems, the Euler ϕ-function, linear congruences, the Chinese remainder theorem, polynomial congruences of degree greater than one, Lagrange's and Wilson's theorems, quadratic reciprocity, and primitive roots. Chapter 6 deals with multiplicative number-theoretic functions, perfect numbers, the Möbius inversion formula, the greatest integer function, and some miscellaneous topics.

The book contains a large number of historical references and, where possible, the theorems have been attributed to their discoverers. There are some 300 exercises to many of which there are detailed answers in the back of the book. The index appears adequate though perhaps a bit abbreviated. The choice of topics for a course in this subject is notoriously idiosyncratic; and though most of the topics selected here would meet with general approval, one does rather question the inclusion of some of the notions in the last three sections of Chapter 6. These notions do seem somewhat inappropriate for the intended audience. Despite these minor matters, the book is most attractive and should serve its intended audience well. Chapter 1, with its careful explanation of mathematical induction and a proof of the equivalence of well-ordering and mathematical induction, is particularly nicely done. The format of the book is most pleasant and there appear to be a minimum of typographical errors.

J. D. BAUM, Oberlin College

Real Numbers. By Stefan Drobot. Prentice-Hall, Englewood Cliffs, N. J., 1964. 102 pp. $3.95.

The content of this book is based on a series of lectures for a summer program for high school teachers and is deserving of the wider audience it should reach in its present form.

By means of careful organization and selectivity, and without sacrifice of clarity or historical background, Drobot is able to proceed fairly quickly through brief chapters on the axiomatic foundations for the real numbers and digital representations (with emphasis on the continued fraction expansion) to the third chapter, which is both the longest and most interesting in the book. Here the discussion centers around the question of approximation of real numbers by rationals. After the motivational background has been prepared a definition is given for the idea of best approximation, followed by proofs of the Hurwitz-Borel theorem and the existence of Liouville numbers as well as several related topics. The book concludes with a fourth chapter consisting of a brief discussion of cardinality and Lebesgue measure and including a proof that the set of continued fractions with bounded digits has measure zero.

The presentation is essentially self-contained and the book is both accessible and appealing to an able undergraduate or teacher who is interested in seeing some significant and fascinating mathematics requiring no extensive formal background. Equally important, perhaps, is that several times the author shows how the results obtained here lead quite naturally to related problems as yet unsolved. The conscientious reader should emerge with a better understanding of the nature and method of mathematical research; it is conceivable that he might be led to try it for himself.

JOHN SCHUE, Macalester College

Vision in Elementary Mathematics. By Walter Warwick Sawyer. Penguin Books, Baltimore, Maryland, 1964. 346 pp. $1.25 (paper).

The author indicates in the preface that this book is designed to give background and assistance to the teacher of today who has had little mathematical training but is called upon to teach arithmetic and mathematics. As a result this volume, which is the first volume of a larger work, deals with very basic concepts in mathematics. It is refreshing, however, to find these concepts presented in a manner which should be clear and understandable even to those who may have had little training or bad training in mathematics. In many cases the introduction of concepts is by means of pictures. Such items as "casting out nines" are introduced at an early point, and, better yet, introduced with the logic presented in rather complete form. In fact, variations of the method of casting out nines are presented for other divisors.

Throughout the book there is the theme of looking for patterns in mathematics, be it in arithmetic, algebra, or geometry. Algebra and geometry are discussed together where this is appropriate. An attempt is made to present mathematics as a logical whole rather than in a fragmented form.

There are some who will quarrel with the book in view of the fact that terms such as "commutative" are not mentioned early in the text. The concepts are

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The format is good and there are numerous illustrations throughout the book. The most regrettable omission is a good index. It may well be that the author foresees including an index at the end of the final volume in this series. It would be more helpful if an index had been included in each volume. The book is remarkably free of misprints.

The person who has had some background in mathematics is likely to find new light shed on some items as he reads this book. On the other hand the person with little or no mathematical background should find this book within his comprehension.

R. L. WILSON, Ohio Wesleyan University

Recreations in the Theory of Numbers—The Queen of Mathematics Entertains. By Albert H. Beiler. Dover, New York, 1964. xvi+349 pages. Paperbound $2.00.

This original Dover Book is attractive, well-planned, and well executed. It seems to be a labor of love by Mr. Albert H. Beiler, a Fellow and life member of the Institute of Electrical and Electronic Engineers.

He tells us in the preface of *Recreations* that he was first attracted to the recreational aspects of the theory of numbers in his own student days. He must have begun early to collect the material for this book, for the extensive bibliographies with which most chapters end comb the literature of the late nineteenth and early twentieth centuries. (One minor criticism is that there are comparatively few references to the literature after 1950.)

Mr. Beiler gives a good exposition of many topics from number theory, together with problems that led to the theory or that have evolved from it. It provides the undergraduate student an opportunity to experience something of the joy that rewards the man who makes discoveries and solves problems in more advanced mathematics.

The book does not replace Ball's *Mathematical Recreations and Essays*, the book that first excited Mr. Beiler's interest in number theory, but it will serve as a worthy companion to it. Another volume that may well join these two on the shelves of a college or secondary school library is *Mathematical Games and Pastimes* translated from the Russian of A. P. Domoryad and published by the Macmillan Company in 1964. The latter book has an interesting (but far less extensive) bibliography with many Russian references, and contains far less theory than does Mr. Beiler's book. I prefer this book: *Recreations in the Theory of Numbers—The Queen of Mathematics Entertains.*

SISTER M. PHILIP STEELE, Rosary College

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PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROBLEMS

621. *Proposed by D. Rameshwar Rao, Osmania University, Secunderabad, India.*

If all three roots of the cubic

$$x^3 + px^2 + qx + r = 0$$

are real, then show that

(1) $p^2 \geqq 3q$

(2) at least one of the roots is less than or equal to

$$\frac{2(p^2 - 3q)^{1/2} - p}{3}.$$

622. *Proposed by Charles W. Trigg, San Diego, California.*

In various lands, there are discothèques with the name *Whisky A-Go-Go*. In the name of the Caribbean one,

$$RUM = AGO + GO,$$

each letter uniquely represents a digit in the scale of six. Rock "*n*" roll out the solution.

623. *Proposed by K. S. Williams, University of Toronto.*

Show that if a is any fixed integer of the form $4b^2+1$, then every integer can be put in the form $x^2+y^2-az^2$.

624. *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Show that a sufficient condition for a sphere to exist which intersects each of four given spheres in a great circle is that the centers of the four given spheres be noncoplanar.

625. *Proposed by Roy Feinman, Rutgers University.*

Consider n independent events. Let their probabilities of occurring be $(\frac{1}{2})^n$, i.e., 1/2, 1/4, $\cdots$, $1/2^n$. What is the limiting value of the probability that at least one of them occurs, as $n \to \infty$?

626. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College.*

In triangle ABC let D, E and F be any points on sides AB, BC and CA respectively. Let G be the point of intersection of AE and DF. Prove that

$$\frac{DG}{GF} = \frac{AD}{AF} \cdot \frac{BE}{CE} \cdot \frac{AC}{AB}.$$

627. *Proposed by Harry W. Hickey, Arlington, Virginia.*

What is the remainder on division of googolplex by 7? ("Googolplex" $=10^g$ where $g=$ "googol" $=10^{100}$.)

SOLUTIONS

Late Solutions

P. N. Bajaj, Western Reserve University: **597**, **599**; *Buster Dunsmore*: **599**; *E. S. Langford, U. S. Naval Postgraduate School, Monterey, California*: **589**, **590**, **591**, **592**; *Lieselotte Miller, Georgia Institute of Technology*: **587**, **590**; *Stanley Rabinowitz, Far Rockaway, New York*: **588**; *George T. Spisak, Broome Technical Community College, New York*: **593**.

Comment on Problem 587

587. [May, 1965, and January, 1966] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove the following inequality

$$\left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \cos^4 \frac{1}{2}\theta < 1, \qquad (-\pi < \theta < +\pi).$$

Comment by the proposer.

The given inequality is equivalent to

$$\left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \left(\frac{1 + \cos\theta}{2}\right)^2 < 1.$$

Now consider the cycloid

$$x = \theta + \sin\theta$$

$$y = 1 + \cos\theta$$

and the ellipse

$$\frac{x^2}{\pi} + \frac{y^2}{4} = 1.$$

They have common origin and equal diameters. The two curves have points in common at the three vertices. We can prove that at the neighborhoods of these points the cycloid lies inside the ellipse. Since their concavity is in the same direction, the cycloid lies wholly inside the ellipse except at the three points. The above inequality is the analytical interpretation for the property just mentioned.

Related Triangles

600. [November, 1965] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If the area of a triangle ABC is S and the areas of the in- and ex-contact triangles are T, T_a, T_b, T_c, then show that

(1) $$T_a + T_b + T_c - T = 2S$$

(2) $$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = 0.$$

Solution by the proposer.

Let I be the incenter and DEF be the in-contact triangle of ABC and let R, r be circumradius and inradius respectively. Then

$$IEF/S = \tfrac{1}{2}r^2 \sin(\pi - A)/(\tfrac{1}{2}bc \sin A)$$
$$= r^2/bc = ar^2/abc = ar^2/4RS$$

or

$$IEF = ar^2/4R$$

and similarly

$$IFD = br^2/4R$$
$$IDE = cr^2/4R.$$

Thus

$$T = IEF + IFD + IDE = (a + b + c)r^2/4R = 2ur\cdot r/4R = S/2R$$

and similarly

$$T_a = Sr_a/2R, \qquad T_b = Sr_b/2R, \qquad T_c = Sr_c/2R.$$

We then have

(1) $$T_a + T_b + T_c - T = S(r_a + r_b + r_c)/2R - Sr/2R$$
$$= S(4R + r - r)/2R = 2S$$

(2) $$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = 2R(1/r_a + 1/r_b + 1/r_c - 1/r)/S = 0.$$

Also solved by P. N. Bajaj, Western Reserve University; Stanley Rabinowitz, Far Rockaway, New York; G. L. N. Rao, J. C. College, Jamshedpur, India.

Euler's Function

601. [November, 1965] *Proposed by David Singmaster, University of California, Berkeley.*

Let ϕ be Euler's function. It is well known that $\phi(x)=14$ has no solutions and that 14 is the smallest even number with this property. Show that there are infinitely many integers m such that the equation $\phi(x)=2m$ has no solutions.

Solution by Douglas Lind, University of Virginia.

THEOREM. *If m is a prime such that $2m+1$ is composite, then $\phi(x)=2m$ has no solutions.*

Proof. We may assume $m>3$. If

$$x = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$$

p_j's distinct primes, $a_j\geqq 1$, then

$$\phi(x) = (p_1-1)(p_2-1)\cdots(p_k-1)p_1^{a_1-1}p_2^{a_2-1}\cdots p_k^{a_k-1} = 2m.$$

Now $2m\mid(p_j-1)$ implies $2m+1=p_j$, a contradiction. Also, $m\nmid(p_j-1)$ since m is a prime >3. Thus $m\mid p_j^{a_j-1}$ for some j, which implies by primality of m that $m=p_j$, so that $\phi(x)\geqq(p_j-1)p_j>2p_j=2m$, this contradiction establishing the theorem.

By the Dirichlet Theorem, there are an infinite number of primes in the sequence $\{3n+1\}$, and $2(3n+1)+1=3(2n+1)$ is composite. Thus, there are an infinite number of primes satisfying the conditions of the theorem, thereby establishing the result of the problem.

Also solved by Erwin Just, Bronx Community College; Charles C. Lindner, Coker College, South Carolina; Harsh Pittie, Swarthmore College; Stanley Rabinowitz, Far Rockaway, New York; D. L. Silverman, Beverly Hills, California; A. M. Vaidya, Bombay, India; John Waddington, Levack, Ont., Canada; K. L. Yocom, South Dakota State University; and the proposer.

Expanding $(x+\lambda)^n$

602. [November, 1965] *Proposed by Bruce W. King, SUNY at Buffalo, New York.*

Show that

$$\sum_{i=0}^{n-2}\binom{n}{i}(-x)^{n-i}(x+\lambda)^i(n-i-1)$$

gives all but the last two terms of the expansion of $(x+\lambda)^n$.

I. *Solution by Eldon Hansen, Lockheed Missiles and Space Company, Palo Alto, California.*

Let

$$S = \sum_{i=0}^{n-2}\binom{n}{i}(-x)^{n-i}(x+\lambda)^i(n-i-1).$$

Since

$$\frac{d}{d\lambda}(x+\lambda)^{i-n+1} = -(n-i-1)(x+\lambda)^{i-n},$$

we have

$$S = -(x+\lambda)^n \frac{d}{d\lambda}\left[(x+\lambda)^{1-n}\sum_{i=0}^{n-2}\binom{n}{i}(-x)^{n-i}(x+\lambda)^i\right].$$

Note that this sum is the first $n-1$ terms of the binomial expansion of

$$[(x+\lambda)-x]^n = \lambda^n$$

and hence

$$\begin{aligned} S &= -(x+\lambda)^n \frac{d}{d\lambda}\left\{(x+\lambda)^{1-n}[\lambda^n + nx(x+\lambda)^{n-1} - (x+\lambda)^n]\right\} \\ &= (x+\lambda)^n - \lambda^n - nx\lambda^{n-1}. \end{aligned}$$

This is the desired result since $nx\lambda^{n-1}$ and λ^n are the last two terms in the binomial expansion of $(x+\lambda)^n$.

II. *Solution by the proposer.*

Let

$$A_n = [a_{ij}]_n$$

be the matrix defined by:

$$a_{ij} = \begin{cases} x & \text{if } i \neq j \\ x+\lambda & \text{if } i = j. \end{cases}$$

Let

$$B_n = A_n - (x+\lambda)I_n,$$

where I_n is the identity matrix of order n.

It is well known that $\det A_n = nx\lambda^{n-1}+\lambda^n$, the last two terms of the binomial expansion of $(x+\lambda)^n$. (See, for example, Franz E. Hohn's *Elementary Matrix Algebra*, on page 71, exercise 29.)

Hohn also gives (on pages 87 and 88) a diagonal expansion of the determinant of a matrix. Using this result, one obtains

$$\det A_n = (x+\lambda)^n + \sum_{i=0}^{n-2}\binom{n}{i}(x+\lambda)^i \det B_{n-i}. \tag{1}$$

But $\det B_{n-i} = (-x)^{n-i-1}(n-i-1)x = -(-x)^{n-i}(n-i-1)$.

Using this last expression for $\det B_{n-i}$ in (1) and equating the two expressions for $\det A_n$, we obtain the desired result.

Also solved by Jack M. Elkin, Brooklyn Polytechnic Institute; Beatriz Margolis, University of Maryland; Stanley Rabinowitz, Far Rockaway, New York; G. P. Speck, U. S. Naval Academy; K. L. Yocom, South Dakota State University; and the proposer (a second solution).

Inverse Points

603. [November, 1965] *Proposed by C. S. Venkataraman, Trichur, South India.*

The center of a circle is A, B a point outside it, and C a point on its circumference. BD and BE are tangents and the perpendicular at C to BC meets AD, AE in F, G respectively. If FM and GN are the perpendiculars to AB, prove:

(a) Triangles ABF and AGB are similar

(b) M and N are inverse points with respect to the circle.

Solution by the proposer.

Clearly angle BAF = angle BAE. Using the cyclic quadrilaterals $FDCB$ and $GEBC$, we can show that angle ABF = angle AGB or angle AFB = angle ABG. In either case, the triangles ABF and AGB are similar.

This fact yields the relation

$$AB^2 = AF \cdot AG. \tag{1}$$

Using similar triangles AGN and AEH (where H is the intersection of AB and DE), we have

$$\frac{AN}{AH} = \frac{AG}{AE} = \frac{AG}{r} \quad (r \text{ the radius of the circle}). \tag{2}$$

From the similar triangles AMF and AHD, we have

$$\frac{AM}{AH} = \frac{AF}{r}. \tag{3}$$

Multiplying (2) and (3) gives

$$\frac{AM \cdot AN}{AH^2} = \frac{AF \cdot AG}{r^2} = \frac{AB^2}{r^2}.$$

Therefore

$$\begin{aligned} AM \cdot AN &= \frac{(AH \cdot AB)^2}{r^2} \\ &= \frac{(r^2)^2}{r^2} \\ &= r^2 \end{aligned}$$

and thus M and N are inverse points with respect to the circle.

Also solved by Stanley Rabinowitz, Far Rockaway, New York, and John Waddington, Levack, Ont., Canada.

A Commutative Ring

604. [November, 1965] *Proposed by Michael Gemignani, University of Notre Dame.*

Let R be a commutative ring with unity, and End R^+ be the ring of additive endomorphisms of R.

(a) Prove that End R^+ is a ring isomorphic to R if and only if it is commutative.

(b) Does the conclusion of (a) follow if we do not require R to have a unity?

Solution by Harsh Pittie, Swarthmore College.

(a) If End $R^+ \cong R$, it must be commutative.

Conversely, let $x, y \in R, f \in$ End R^+, and define $y(x) = yx$ so that $y \in$ End R^+. Then we have

$$f(yx) = f(y(x)) = y(f(x)) = yf(x)$$

and therefore

$$f(y) = yf(e)$$

in particular. Finally, $f \rightarrow f(e)$ is an isomorphism of End R^+ onto R.

(b) Obviously, since End R^+ always has a unity, R cannot be isomorphic to it unless it has one, too!

However, R may be isomorphically *imbedded* in End R^+. Adjoin a unity to R and call the resulting ring S; clearly,

$$\text{End } S^+ \cong \text{End } R^+.$$

But by (a),

$$S \cong \text{End } S^+.$$

Note. Under the suppositions of (a), we may identify Hom(R, R) the set of all ring-endomorphisms of R as the set of idempotents in R.

Proof. Hom$(R, R) \subseteq$ End R^+ and if we identify End R^+ with R—since they are isomorphic—we get Hom$(R, R) \subseteq R$.

Now, let $x \in$ Hom(R, R); then

$$x(e \cdot e) = x(e)x(e)$$

so that

$$x = x^2.$$

Also, if x is an idempotent in R, then

$$x(yz) = x^2(yz) = xyxz = x(y)x(z)$$

and, of course,

$$x(y + z) = xy + xz.$$

Query. How is Hom(R, R) related to R under the suppositions of part (b)?

Also solved by Charles R. Coniff, Wisconsin State University, LaCrosse, Wisconsin; J. F. Leetch, Bowling Green State University, Ohio; Charles C. Lindner, Coker College, South Carolina; and the proposer.

Bridging a Moat

605. [November, 1965] *Proposed by Sam Newman, Atlantic City, New Jersey.*

A square island is surrounded by a square moat of water of width x feet. Explorers wishing to get to the island have available only two boards of length y feet ($y<x$) of negligible width. What is the largest width the moat can be for them to reach the island using these boards?

Solution by Howard Marston, Principia Upper School, St. Louis, Missouri.

We wish to get some point of the first board as close as possible to some point on the edge of the island yet support the board at both ends on the outer rim of the moat. So we place the first board of length y diagonally across a corner. For a moat of maximum width x, the midpoint of this first board is a distance y from a corner of the island. The first board forms with the corner of the moat an isosceles triangle with legs $y\sqrt{2}/2$ feet long. Consider a similar triangle formed by this corner and a line through the corner of the island. Its legs are $2x$ feet long. The resulting isosceles trapezoid with altitude y has legs $y\sqrt{2}$ feet long. Hence

$$2x = \frac{y\sqrt{2}}{2} + y\sqrt{2};$$

so

$$x = \tfrac{3}{4}y\sqrt{2}.$$

Consequently, the maximum width of the moat is $\frac{3}{4}\sqrt{2}$ times the length of each board.

Also solved by Maxey Brooke, Sweeney, Texas; Frank M. Eccles, Phillips Academy, Andover' Massachusetts; Jack M. Elkin, Brooklyn Polytechnic Institute; Michael Goldberg, Washington, D.C.; Larry Hoehn, Perryville, Missouri; J. F. Leetch, Bowling Green State University, Ohio; Stanley Rabinowitz, Far Rockaway, New York; Charles W. Trigg, San Diego, California; K. L. Yocom, South Dakota University; and the Proposer.

Goldberg found a more complicated version of a similar problem in *Canterbury Puzzles* by Dudeney, Problem No. 54, pp. 56–57. One incorrect solution was received.

A Definite Integral

606. [November, 1965] *Proposed by Gilbert Labelle, Université de Montréal, Canada.*

Evaluate

$$\int_0^1 \frac{1-x^2}{1+x^2}\cdot\frac{dx}{\sqrt{\{1+x^4\}}}\,.$$

Solution by Clyde A. Bridger, Springfield Junior College, Illinois.

The identity

$$1 + z^4 = (1 - z^2)^2 + 2z^2$$

suggests putting

$$2z^2 = (1 + z^4)t^2.$$

Then

$$(1 - z^2)^2 = (1 + z^4)(1 - t^2)$$

and

$$(1 + z^2)^2 = (1 + z^4)(1 + t^2).$$

After taking the logarithmic derivative of the last expression, collecting terms, and factoring, the result is

$$\frac{1 - z^2}{1 + z^2} \cdot \frac{dz}{\sqrt{1 + z^4}} \cdot \frac{2z^4}{\sqrt{1 + z^4}} = \frac{t\,dt}{1 + t^2}.$$

Now, when $z=0$, $t=0$ and when $z=1$, $t=1$. Whence

$$I = \frac{1}{\sqrt{2}} \int_0^1 \frac{dt}{1 + t^2} = \frac{\pi\sqrt{2}}{8}.$$

Comment. Set $t^m = \rho^m/a^m$ and write $\rho^m = a^m \cos m\phi$, where ρ and ϕ are polar coordinates. Then

$$I = \int_0^{\pi/2m} \tan \frac{m\phi}{2} \cdot ds,$$

where ds is the element of arc for the chosen curve of the family. The substitution

$$2z^2/(1 + z^4) = \cos m\phi = t^m$$

leads to

$$a\,dt/(1 - t^m).$$

When $m=2$, the proposed integral whose solution is above is obtained. (When $m=1$, the curve is a circle.) Without the rationalizing factor $(1-z^2)/(1+z^2)$ when $m=2$, I is an elliptic integral of the first kind whose solution is given in Problem 564.

Also solved by P. N. Bajaj, Western Reserve University; Cornelio Binoya, Jr., Central Luzon State University, Philippines; L. Carlitz, Duke University; Charles R. Conniff, Wisconsin State University, LaCrosse, Wisconsin; R. J. Cormier, Northern Illinois University; Robert V. Esperti and E. S. Wilbarger, Jr., (Jointly), Santa Barbara, California; Richard Feldman, Lycoming College, Pennsylvania; Henry E. Fettis, Wright-Patterson A.F.B., Ohio; Dick Wick Hall, Harpur College, *New York; Stanley Rabinowitz, Far Rockaway, New York; G. L. N. Rao, J. C. College, Jamshedpur, India; Paul D. Thomas, U. S. Naval Oceanographic Office, Suitland, Maryland; L. V. Venkataraman, University of New Hampshire; John Waddington, Levack, Ont., Canada; K. L. Yocom, South Dakota State University; and the proposer. Two incorrect solutions were received.*

Other substitutions included

$$y = (1 - x^2)/(1 + x^2) \qquad [\textit{Carlitz}]$$
$$x^2 = (1 - \sin^{1/2}\theta)/(1 + \sin^{1/2}\theta) \qquad [\textit{Cormier}]$$
$$x = \sin\phi/(1 + \cos\phi) \qquad [\textit{Fettis}]$$
$$U = \text{arc}\tan x \qquad [\textit{Hall}]$$

and

$$x = \{[(1 + v^2)^{1/4} - 1]/[(1 + v^2)^{1/4} + 1]\}^{1/2} \qquad [\textit{Thomas}]$$

Jacques Allard, University of Sherbrooke, Canada, found the problem in Ginzburg, *Calculus*, page 170, Problem No. 816.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q383. Determine all the numbers which are both Fibonacci (1, 1, 2, 3, 5, 8, · · ·) and Lucas (1, 3, 4, 7, 11, 18, · · ·).

[*Submitted by David L. Silverman*]

Q384. Let D, E, F be the midpoints of the arcs BC, CA, AB of the circumcircle of an acute triangle ABC and let L, M, N be the feet of the perpendiculars from D, E, F, respectively, upon BC, CA, AB. If r is the inradius of triangle ABC, show that

$$DL + EM + FN + r$$

is equal to the circumdiameter $2R$.

[*Submitted by Leon Bankoff*]

Q385. Show that all the points at infinity lie on exactly one straight line.

[*Submitted by Stanley Rabinowitz*]

Q386. Show that if p_1 and p_2 are successive odd primes and if $p_1+p_2=2q$, then q is composite.

[*Submitted by John D. Baum*]

Q387. Plant an orchard of 20 trees in 10 rows of 5 trees each.

[*Submitted by Charles E. Maley*]

(Answers on page 160)

References

1. M. C. Gemignani, On finite subsets of the plane and simple closed polygonal paths, this Magazine, 39 (1966) 38–42.

2. L. Quintas and F. Supnick, On some properties of shortest Hamiltonian circuits, Amer. Math. Monthly, 72 (1965) 977–980.

ANSWERS

A383. $F_4 < L_3 < F_5 < L_4 < F_6$. The Lucas-Fibonacci recursion guarantees that $F_6 < L_5 < F_7 < \cdots$. By induction, the two sequences remain strictly interleaved. Hence, 1 and 3 are the only numbers common to both sequences.

A384. Extend DL, EM, FN to meet at the circumcenter O. Since

$$OL + OM + ON = R + r,$$

we have

$$3R - (DL + EM + FN) = R + r$$

or

$$2R = DL + EM + FN + r.$$

A385. Let P, Q, R be any three distinct points at infinity. Choose real lines p, q, r to pass through these points, respectively. These lines determine a real triangle ABC. Now P, Q, R divide the sides of triangle ABC in the ratio -1. $(-1)(-1)(-1) = -1$. Hence P, Q, R are collinear by the theorem of Menelaus.

A386. We have

$$p_1 < \frac{p_1 + p_2}{2} < p_2$$

and

$$q = \frac{p_1 + p_2}{2}\,.$$

Thus, since p_1 and p_2 are successive primes, q is not a prime.

A387. Plant five trees forming the pentagon 1 2 3 4 5. Then plant five at these intersections of pairs of opposite sides, as (1 2) (34), $\cdots$, (5 2) (1 3). Plant five more at the intersections of pairs of diagonals, as (1 3) (2 4), $\cdots$, (5 2) (1 3), and plant the last five at the intersections of side-diagonal pairs, as (1 2) (3 5), $\cdots$, (5 1) (2 4). The first of the four pentagons must be such that none of the 15 pairs of lines are parallel, so that it can not, for example, be regular on any finite form.

(Quickies on page 196)